Various Attractors, Coexisting Attractors, Antimonotonicity in a Simple Four-order Memristive Twin-T Oscillator

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Received (to be inserted by publisher)

By replacing the resistor in a Twin-T network with generalized flux-controlled memristor, this paper proposes a simple four-order memristive Twin-T oscillator. Rich dynamical behaviors can be triggered in the dynamical system. The most striking feature is that this system has various periodic orbits and various chaotic attractors by adjusting parameter $b$. At the same time, coexisting attractors and antimonotonicity (Especially, two full Feigenbaum remerging trees in series are observed in autonomous chaotic systems.) are also detected, and their dynamical features are analyzed by phase portraits, Lyapunov exponents, bifurcation diagram and basin of attraction. Moreover, hardware experiments on a breadboard are carried out. Experimental measurements are accord with the simulation results. At last, we have developed a multi-channel random bit generator for encryption application. Numerical results illustrate the usefulness of the random bit generator.

Keywords: Twin-T oscillator; memristor; antimonotonicity; various attractors; random bit generator; encryption

1. Introduction

Memristor, as the fourth basic circuit element, postulated in 1971 is a nonlinear two-terminal electronic element [Chua et al., 1971]. Generalization memristive systems have been proposed by Prof. Chua in 1976, and the most salient feature of memristive systems is its zero-crossing property [Chua et al., 1976]. Since the physical memristor has been fabricated by Hewlett-Packard Lab in 2008 [Strukov et al., 2008], it has attracted much attention of researchers. Due to its non-volatility, nano-size, and low power consumption, memristor can be applied in various fields, such as nonvolatile memory [Strukov, 2016], neural networks [Soudry et al., 2015];[Duan et al., 2015], nonlinear chaotic circuits [Zhou et al., 2016];[Wu et al., 2016];[Yuan et al., 2016];[Wang et al., 2017];[Corinto et al., 2017];[Ma et al., 2017], and so on.

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Oscillator is frequently used as signal generator. It is thought that chaotic signal generation with wide frequency spectrum has potential application for secure communication and image encryption [Ren et al., 2017]; [Lin et al., 2016]; [Li et al., 2017]. In addition, the long-term unpredictability of chaotic signal makes these systems well suited for true random bit generation [Yalcin et al., 2004]. In robotics, chaotic signals are being used in neural control networks or as chaotic path generators [Li et al., 2015]. In all these applications, they need exists for flexible, preferably, and low-cost chaotic circuits. A large number of chaotic oscillators were reported [Morgui et al., 1995]; [Namajunas et al., 1996]; [In et al., 2010]; [Elwakil et al., 2000]; [Keuninckx et al., 2015]. By substituting nonlinear element or linear element with memristor in classic oscillators, memristive oscillators with complex dynamical behaviors are recently discussed [In et al., 2011]; [Bao et al., 2015]; [Xu et al., 2016]; [Chen et al., 2015]; [Yu et al., 2014]; [Wu et al., 2016]; [Bao et al., 2017a].

By replacing the Chua diode with memristor in Chua’s circuit consisting of inductor, the rich varieties of bifurcation structures are analyzed [Bao et al., 2015]; [Xu et al., 2016]; [Chen et al., 2015]. The Wien-bridge nonlinear oscillator composed of a Wien-bridge oscillator, memristor, and LC network is proposed, and there exist complex nonlinear phenomena including periodic oscillations, chaotic oscillations and fast-slow effects [Yu et al., 2014]. In 2016, a simple memristive Wien-bridge oscillator is constituted by replacing the resistor of parallel RC network with generalized memristor [Wu et al., 2016]; however, the generalized memristor contains LC network. The above memristive oscillators all include inductor [In et al., 2011]; [Bao et al., 2015]; [Xu et al., 2016]; [Chen et al., 2015]; [Yu et al., 2014]; [Wu et al., 2016]. It is well known that the presence of an inductor with parasitic equivalent series resistor makes circuit hardware bulky, unsuitable for IC design, less robust, etc. Later, by linearly coupling a parallel memristor and capacitor filter to a single amplifier biquad-based inductor-free Chua’s circuit with an ideal memristor [Bao et al., 2016], or by replacing a nonlinear resistor and a linear resistor with two different memristors in an active band pass filter-based Chua’s circuit [Bao et al., 2017a], extreme multistability and other complex dynamical behaviors are discussed.

In 2014, by substituting the resistor with memristor, the memristive relaxation Oscillator provides novel and steady oscillating behaviors [Yu et al., 2014], however, it can not generate chaotic signal. In 2010, chaos in a Twin-T circuit is described, and the circuit is very simple, requiring just one op-amp stage. However, the circuit must be driven by a low frequency input-signal [In et al., 2010]. A five-order inductor-free nonlinear oscillator composed of a Twin-T oscillator, memristor, and RC network is designed [Li et al., 2013].

In this paper, a generalized memristor is used to directly replace the resistor in a Twin-T oscillator. Compared with the reference [In et al., 2010], the proposed chaotic circuit doesn’t need extra input-signal. At the same time, compared with the reference [Li et al., 2013], the proposed Twin-T oscillator with memristor, keeping original oscillator circuit structure, doesn’t need to add extra RC network. Moreover, it has a relatively simpler electronic structure (absence of RC network) and complex dynamical behaviors. Compared to former reported [In et al., 2010]; [Elwakil et al., 2000]; [Li et al., 2013], the new memristive Twin-T system has various periodic orbits and various chaotic attractors by adjusting parameter b. Only adjusting the Resistor R2, therefore, it is convenient and easy in different oscillator application. Moreover, it has also many interesting properties such as series antimonotonicity, and coexisting attractors. Although the phenomenon of antimonotonicity is observed in many circuits, as far as 1 known, antimonotonicity (two full Feigenbaum remerging trees in series) is less observed in the autonomous chaotic systems.

The paper is organized as follows. In Sec. 2, the circuit structure of the memristive Twin-T oscillator is depicted, and a suitable mathematical model is obtained. In Sec. 3, some basic properties of the mathematical model such as dissipativity, equilibrium, and stability are analyzed. In Sec. 4, simulation analyses of the dynamical system are investigated by phase portraits, bifurcation diagram and Lyapunov Exponents. Many various periodic attractors, various chaotic attractors, coexisting attractors (for the same parameters settings) and antimonotonicity are discussed. In Sec. 5, hardware experiments on a breadboard are carried out. In Sec. 6, a random bit generator design with the memristive Twin-T chaotic system is proposed, the three outputs of the memristive Twin-T oscillator all successfully pass all the tests NITS-800-22. The conclusive remarks are finally drawn in Sec. 7.
2. Circuit Description

By substituting the resistor in Twin-T oscillator with memristor, the new proposed memristive Twin-T oscillator is depicted in Fig. 1.

Fig. 1. The proposed Twin-T oscillator with memristor

Due to the memristor is not commercially available, various kinds of memristive emulators have been proposed for studying the application circuits of memristor. According to the form of generalized voltage-controlled memristor defined by Chua in [Chua et al., 1976], the emulator proposed by Bao and co-workers [Bao et al., 2017b] only has eight elements and can be easily physically implemented, as shown in Fig. 2, where \( v \) and \( i \) stand for the voltage and the current at the input port of memristor, respectively. \( V_\varphi \) denotes the voltage across the integral capacitor \( C_\varphi \), and \( g \) denotes the scale factor of multiplier. The voltage-current relationship of the memristor emulator is modeled as

\[
i = W(V_\varphi)v = \frac{v - g\varphi(gV^2_\varphi)}{R_\varphi} = \frac{1}{R_\varphi}(1 - g^2V^2_\varphi)v
\]

where \( W(V_\varphi) \) is a continuous linear memductance function related to \( V_\varphi \), which can be given by

\[
W(V_\varphi) = \frac{1}{R_\varphi}(1 - g^2V^2_\varphi)
\]

Fig. 2. The memristor emulator

The circuit parameters in Figs.1 and 2 are given in Table 1. Due to absence of RC network, the new proposed memristive Twin-T chaotic circuit is a four-order nonlinear circuit. According to the constitutive relation of memristor and the KVL equations of Twin-T chaotic circuit, the dynamical equations of
Table 1. Element parameters of memristive Twin-T oscillator

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Significations</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1, C_2, C_3, C_\phi$</td>
<td>Capacitance</td>
<td>10 nF</td>
</tr>
<tr>
<td>$R_\phi$</td>
<td>Resistance</td>
<td>13.3 kΩ</td>
</tr>
<tr>
<td>$R_1$</td>
<td>Resistance</td>
<td>28.57 kΩ</td>
</tr>
<tr>
<td>$R_2$</td>
<td>Resistance</td>
<td>285.7 kΩ</td>
</tr>
<tr>
<td>$R_{m1}$</td>
<td>Resistance</td>
<td>10 kΩ</td>
</tr>
<tr>
<td>$R_{m2}$</td>
<td>Resistance</td>
<td>100 kΩ</td>
</tr>
<tr>
<td>$R_A$</td>
<td>Resistance</td>
<td>1 kΩ</td>
</tr>
<tr>
<td>$R_D$</td>
<td>Resistance</td>
<td>10 kΩ</td>
</tr>
</tbody>
</table>

The memristive Twin-T chaotic circuit in Fig. 1 can be described as

$$
\begin{align*}
C_1 \frac{dV_1}{dt} &= (1 + k) \frac{1}{R_\phi} (1 - g^2 V_2^2) V_2 - \frac{2 V_3 (1 + k)}{R_2} + \frac{2 k V_1}{R_2} \\
C_2 \frac{dV_2}{dt} &= k \frac{1}{R_\phi} (1 - g^2 V_2^2) V_2 - \frac{V_3 (1 + 2 k)}{R_2} + \frac{(2 k + 1) k V_1}{R_2 (1 + k)} \\
C_3 \frac{dV_3}{dt} &= \frac{V_1}{R_1} + \frac{k V_1}{R_2} - \frac{V_3}{R_1} - \frac{V_3}{R_2} \\
C_\phi \frac{dV_\phi}{dt} &= - \frac{V_2}{R_{m1}} - \frac{V_\phi}{R_{m2}}
\end{align*}
$$

where $k = R_A/R_B$, $V_1, V_2, V_3$ and $V_\phi$ are four node voltages standing for four state variables.

Let $t = \tau RC$, where $\tau$ is the dimensionless time. The parameters are expressed as follows:

$$
x = V_1, y = V_2, z = V_3, u = V_\phi, C_1 = C_2 = C_3 = C, a = R/R_\phi, \\
b = R/R_2, c = R/R_1, m = RC/(R_{m1} C_\phi), n = RC/(R_{m2} C_\phi)
$$

Therefore, the dimensionless equations can be described as follows:

$$
\begin{align*}
\dot{x} &= (1 + k) a (1 - g^2 u^2) y - 2 (1 + k) b z + 2 k b x \\
\dot{y} &= k a (1 - g^2 u^2) y - (1 + 2 k) b z + (2 k + 1) b x / (1 + k) \\
\dot{z} &= c x + b k x / (1 + k) - c z - b z \\
\dot{u} &= - m y - n u
\end{align*}
$$

When $R = 100$ kΩ and $C = 10$ nF, the numerical experiment is carried out with the following dimensionless parameters: $a = 7.5, b = 0.35, c = 3.5, m = 10, n = 1, k = 0.1, g = 0.1$.

3. Dissipativity, equilibrium, and stability

It can be observed obviously that the system (5) is symmetrical about the origin and invariant under the transformation $(x, y, z, u) \rightarrow (-x, -y, -z, -u)$, which indicates if $(x, y, z, u)$ is a solution of (5), then $(-x, -y, -z, -u)$ is also a solution.

To evaluate the dissipativity of system (5), mathematical expression of exponential constrain rate is deduced as

$$
\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{u}}{\partial u} = ka (1 - g^2 u^2) + 2 k b - c - b - n
$$

When $ka < c + b + n - 2 k b$, (6) is negative, i.e., $\Delta V < 0$, implying that the system is dissipative. This means that asymptotic motion settles onto an attractor and each volume containing the system trajectory shrinks to zero at an exponential rate as $t \rightarrow \infty$. 
The equilibria of system (5) can be derived by solving the following equations:
\[
\begin{align*}
(1 + k)a(1 - g^2u^2)y - 2(1 + k)bz + 2kbx &= 0 \\
ka(1 - g^2u^2)y - (1 + 2k)bz + (2k + 1)bky / (1 + k) &= 0 \\
ex + bky / (1 + k) - cz - bz &= 0 \\
-my - au &= 0 
\end{align*}
\] (7)

We can easily observe that the system (5) has three equilibrium points including one zero equilibrium and two symmetric nonzero equilibria, which are described by:
\[
\begin{align*}
P_0 &= (0, 0, 0, 0) \\
P_\pm &= (0, \frac{m}{2}, 0, \pm 1/g)
\end{align*}
\] (8)

By linearizing system (5) at zero equilibrium \( P_0 \), we can obtain the Jacobian matrix.
\[
J_0 = \begin{pmatrix}
2kb & (1 + k)a - 2(1 + k)b & 0 \\
\frac{(2k+1)kb}{1+k} & ka & -(1 + 2k)b \\
\frac{1+k}{c+1+k} & 0 & -(c + b) \\
0 & -m & 0 & -n
\end{pmatrix}
\] (9)

\[
\det(\lambda I - J_0) = (\lambda + n)(\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3) = 0
\]
\[
a_1 = c + b - ak - 2bk \\
a_2 = 2bc - aek - 2ab \\
a_3 = -(4abc + 4ad^2e^2 + 6abc + 2ad^2k + abc)
\] (10)

The coefficients of the cubic polynomial equation of (10) are all nonzero, and the Routh-Hurwitz conditions for the above cubic polynomial are given by
\[
\begin{align*}
a_1 &> 0 \\
a_3 &> 0 \\
a_1a_2 - a_3 &> 0
\end{align*}
\] (11)

It is found that when the parameters of system (5) are all positive, \( a_3 \) is always negative, and the Routh-Hurwitz conditions are not satisfied. When the parameters of system (5) are taken as \( a = 7.5, b = 0.35, c = 3.5, m = 10, n = 1, k = 0.1, \) and \( g = 0.1 \), the eigenvalues at \( P_0 \) are calculated as
\[
P_0 : \lambda_1 = -1, \quad \lambda_{2,3} = 0.4033 \pm 1.4940i, \quad \lambda_4 = -3.8366
\] (12)

which implies that \( P_0 \) is an unstable saddle-focus.

Similarly, the Jacobian matrix at two nonzero equilibria \( P_\pm \) can be deduced as
\[
J_\pm = \begin{pmatrix}
2kb & 0 & 2m(1+k) \\
\frac{(2k+1)kb}{1+k} & 0 & 2km \\
\frac{1+k}{c+1+k} & 0 & 0 \\
0 & -m & 0 & -n
\end{pmatrix}
\] (13)

\[
\det(\lambda I - J_\pm) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0
\]
\[
a_1 = c + n + b - 2bk \\
a_2 = 2bc + 2akn + cn + bn - 2bkn \\
a_3 = 2ackn - 8abc + 8ad^2n + 2bcn \\
a_4 = -(8abcnk^2 + 8ad^2nk^2 + 4abc + 4ad^2nk + 2abc)
\] (14)

The coefficients of the quartic polynomial equation of (14) are all nonzero, and the Routh-Hurwitz conditions for the above quartic polynomial are given by
\[
\begin{align*}
a_1 &> 0 \\
a_4 &> 0 \\
a_1a_2 - a_3 &> 0 \\
a_1(a_3a_2 - a_4a_1) - a_1^2 &> 0
\end{align*}
\] (15)
It is found that when the parameters of system (5) are all positive, \( a_4 \) is always negative, and the Routh-Hurwitz conditions are not satisfied. When the parameters of system (5) are taken as \( a = 7.5, b = 0.35, c = 3.5, m = 10, n = 1, k = 0.1, \) and \( g = 0.1 \), the eigenvalues at \( P_\pm \) are calculated as
\[
P_\pm: \lambda_1 = -3.6932, \quad \lambda_{2,4} = -0.9947 \pm 2.1266i, \quad \lambda_4 = 0.9026
\] (16)
which implies that \( P_\pm \) has an unstable saddle.

4. Numerically simulation of Twin-T oscillator

4.1. Various periodic orbits and various chaotic attractors

The parameter \( b \) (Resistor \( R_2 \)) will be considered as the main bifurcation control parameter in this work, and the others of system (5) are taken as \( a = 7.5, c = 3.5, m = 10, n = 1, k = 0.1, \) and \( g = 0.1 \). Changing the parameter \( b \) from 0.08 to 0.6 and setting the initial conditions \([1 \ 0 \ 0 \ 0\) and \([-1 \ 0 \ 0 \ 0)]\), we can obtain bifurcation diagrams in Fig. 3 (a). First three Lyapunov exponents are shown in Fig. 3(b) when the initial condition is \([1 \ 0 \ 0 \ 0\).

![Bifurcation diagrams and first three Lyapunov exponents](image)

Fig. 3. Bifurcation diagrams and first three Lyapunov exponents: (a) Bifurcation diagrams of \( u \), the red trajectories start from the initial conditions \((-1, 0, 0, 0\), and the blue trajectories emerge from \((1, 0, 0, 0\), (b) First three Lyapunov exponents with respect to \( b \) when the initial condition is \((1 \ 0 \ 0 \ 0\).

From the bifurcation diagrams, we can observe clearly that the system (5) has complex dynamic behaviors. When the parameters of system (5) are taken as \( a = 7.5, c = 3.5, m = 10, n = 1, k = 0.1, \) and \( g = 0.1 \), various attractors can be observed by changing the value of parameter \( b \). Two parameter spaces will be considered in the following.

1. \(0.08 < b < 0.18\)

Changing \( b \) from 0.08 to 0.18, the bifurcation diagrams of \( u \) (the interval of \( b \) is \([0.08, 0.18]\)) are depicted in Fig. 4 (a) and partial enlargement of bifurcation diagrams are depicted in Fig. 4 (b). Along with increase of \( b \) in the interval \([0.08, 0.18]\), the orbits transform from coexisting period-1 to 2 to 4 and to 8 successively. Furthermore, three main periodic windows with period-3, period-7 and period-9 appear in the intervals \([0.1001, 0.1062], [0.1182, 0.1183] \) and \([0.1160, 0.1167]\). The corresponding first three Lyapunov exponents with respect to \( b \) are depicted in Fig. 4 (c) when the initial conditions is \((1, 0, 0, 0\). The typical periodic orbits are shown in Fig. 5 when \( b = 0.09 \), and the typical chaotic attractors are shown in Fig. 6 when \( b = 0.11 \). When in the intervals \([0.093, 0.109]\) and \([0.129, 0.13]\), coexisting bifurcation modes appear, and the coexisting attractors are shown in Fig. 7.

2. \(0.18 < b < 0.6\)

From the bifurcation diagrams in Fig. 3, there exist coexisting period orbits, chaotic attractors, and coexisting chaotic attractors. The typical chaotic attractors which are different from Fig. 6 are shown in Fig. 8 when \( b = 0.3 \). When in the intervals \([0.18, 0.24]\) and \([0.38, 0.6]\), coexisting bifurcation modes appear, and the coexisting attractors are shown in Fig. 9.
Fig. 4. Bifurcation diagrams and first three Lyapunov exponents (the interval of $b$ is $[0.08, 0.18]$): (a) Bifurcation diagrams of $u$, the red trajectories start from the initial conditions $(-1, 0, 0, 0)$, and the blue trajectories emerge from $(1, 0, 0, 0)$, (b) First partial enlargement of (a), (c) First three Lyapunov exponents with respect to $b$ when the initial condition is $(1 0 0 0)$.

Fig. 5. Typical period-1 orbits in four-order memristive oscillator with Twin-T when $b = 0.09$: (a) Phase portrait in the $x - y$ plane, (b) Phase portrait in the $x - z$ plane, (c) Phase portrait in the $x - u$ plane

Fig. 6. Typical chaotic attractors in four-order memristive oscillator with Twin-T when $b = 0.11$: (a) Phase portrait in the $x - y$ plane, (b) Phase portrait in the $x - z$ plane, (c) Phase portrait in the $x - u$ plane

From the Figs. 5-9, this new system has various periodic orbits and various chaotic attractors by adjusting parameter $b$. Rich dynamical behaviors can be triggered in the dynamical system (5).

4.2. Basins of attraction

More information about the coexisting attractors can be obtained by analyzing the basins of attraction of the different attracting sets, which are defined as the set of initial conditions whose trajectories converge to the respective attractor [Li et al., 2013]. In order to understand coexisting attractor, the basin of attraction is discussed. Take the coexisting chaotic attractors in Fig. 7 (f) and Fig. 9 (f) as example,
their basin boundaries are clearly observed in Fig. 10 where cross sections of the basins of attraction for $y(0) = u(0) = 0$. Blue and red zones denote the asymmetric pair of chaotic attractors. Green zones denote unbounded motion. From Fig. 10 (a), it can be seen that there are two coexisting attractors shown in Fig. 7 (f) when $b = 0.125$; however, there exists the third coexisting attractor except that shown in Fig. 9 (f) when $b = 0.24$. In Fig. 10 (b), yellow zones denote the third coexisting attractor when $b = 0.24$. The initial conditions of the third coexisting attractor are obtained in the progress of computing basin of attraction (The coordinates corresponding yellow points denote the initial conditions $x_0$ and $z_0$, respectively), therefore, the third coexisting attractor has strict initial condition. When the initial conditions are $(-0.025125628140704, 0, -1.547738693167337, 0)$, the phase portrait in the $x - u$ plane is shown in Fig. 11 (a). Based on the first three Lyapunov exponents depicted in Fig. 11 (b), the third coexisting attractor when $b = 0.24$ is also chaotic attractor.

4.3. Antimonotonicity

It is well known that periodic orbits can be created and then destroyed via reverse period-doubling bifurcation scenarios as a bifurcation control parameter slowly changing [Dawson et al., 1992]. This type of creation and annihilation of periodic orbits is named as antimonotonicity, and it has been observed in various nonlinear systems such as memristive jerk system [Kengne et al., 2017], the Duffing oscillator
Fig. 9. The coexisting attractors in the x-a plane: (a) period-1 ($b = 0.18$), (b) period-2 ($b = 0.2$), (c) period-4 ($b = 0.201$), (d) chaotic attractor ($b = 0.21$), (e) chaotic attractor ($b = 0.22$), (f) chaotic attractors ($b = 0.24$), (g) chaotic attractor ($b = 0.4$), (h) period-3 ($b = 0.427$), (i) chaotic attractor ($b = 0.43$), (j) period-4 ($b = 0.435$), (k) period-2 ($b = 0.47$), (l) period-1 ($b = 0.55$).

[Srinivasan et al., 2009], Chua circuit [Kocarev et al., 1993], laser system [Chionverakis et al., 2006] and series-parallel LC circuit [Manimehan et al., 2012]. From the bifurcation diagrams in Figs. 3-4, the dynamical behaviors can be described as periodic orbits $\rightarrow$ chaotic orbits $\rightarrow$ periodic orbits $\rightarrow$ periodic orbits $\rightarrow$ chaotic orbits $\rightarrow$ periodic orbits. As far as I known, antimonotonicity (two full Feigenbaum remerging trees in series) is less observed in the autonomous chaotic systems. Also, in order to demonstrate the phenomenon of antimonotonicity in our system (5) obviously, some bifurcation diagrams as parameter $b$ are varied in the interval [0.1, 0.4] for some discrete values of parameter $a$. The simulation results are described in Fig. 12. In light of the graphs in Fig. 12, for $a = 0.6$, two primary bubbles are observed and the branch develops a stable period-4 bubble and period-2 bubble in series at $a = 6.7$. Similarly, we have a period-8
5. Hardware experiments

In order to further research the proposed memristive Twin-T chaotic system shown in Figs. 1-2, hardware experiments on a breadboard containing potentiometer, monolithic ceramic capacitor, op-amp TL082 and multiplier AD633JN are made to verify dynamical behaviors of the memristive Twin-T oscillator. The circuit parameters are selected as $C_2 = C_1 = C_3 = 10 \, \text{nF}$, $R_2 = 13.3 \, \text{k} \Omega$, $R_1 = 28.57 \, \text{k} \Omega$, $R_{m1} = 10 \, \text{k} \Omega$, $R_{m2} = 100 \, \text{k} \Omega$, $R_A = 1 \, \text{k} \Omega$, $R_B = 10 \, \text{k} \Omega$. And DC power supplies are $\pm 15 \, \text{V}$. $R_2$ is a variable resistance. The experimental results are photographed, as shown in Figs. 13 - 16. Compared with the Figs. 6 - 9, the experiment results are accord with simulation results.

By switching on and off the power supply, the coexisting attractors can be obtained. Two coexisting chaotic attractors in the $x - u$ plane when $R_2 = 416.6 \, \text{k} \Omega$ are shown in Fig. 17 (Because there are too many figures of coexisting attractors corresponding Figs. 15 - 16, the case when $b = 0.24$ is selected to verify the existence of coexisting attractors).
Fig. 12. Bifurcation diagrams showing local maxima of the coordinate $u$ in terms of control parameter $b$ for remerging Feigenbaum tree (bubbling): (a) two primary bubbles for $a = 6.6$; (b) period-4 bubble and period-2 bubble in series at $a = 6.7$; (c) period-8 bubble and period-2 bubble in series at $a = 6.705$; (d) enlargement of (c); (e) a Feigenbaum remerging tree and period-2 bubble in series at $a = 6.75$; (f) a Feigenbaum remerging tree and period-4 bubble in series at $a = 6.8$; (g)-(h) two full Feigenbaum remerging trees at $a = 6.9$ and $a = 7$, respectively.

Fig. 13. Experimental measured phase portraits when $R_2 = 333.3 \, k\Omega$, and the scales are $x = 2 \, V/\text{div}$, $y = 1 \, V/\text{div}$ and $u = 5 \, V/\text{div}$.

Fig. 14. Experimental measured phase portraits when $R_2 = 969 \, k\Omega$, and the scales are $x = 2 \, V/\text{div}$, $y = 1 \, V/\text{div}$ and $u = 5 \, V/\text{div}$.
Fig. 15. Experimental measured phase portraits under different $R_2$. The scales are $x = 2$ V/div and $u = 5$ V/div for all pictures: (a) $R_2 = 1$ MΩ; (b) $R_2 = 990$ kΩ; (c) $R_2 = 964.3$ kΩ; (d) $R_2 = 952$ kΩ; (e) $R_2 = 941.6$ kΩ; (f) $R_2 = 800$ kΩ.

Fig. 16. Experimental measured phase portraits under different $R_2$. The scales are $x = 2$ V/div and $u = 5$ V/div for all pictures: (a) $R_2 = 555.5$ kΩ; (b) $R_2 = 500$ kΩ; (c) $R_2 = 497.5$ kΩ; (d) $R_2 = 470.1$ kΩ; (e) $R_2 = 454.5$ kΩ; (f) $R_2 = 416.6$ kΩ; (g) $R_2 = 250$ kΩ; (h) $R_2 = 235.29$ kΩ; (i) $R_2 = 232.55$ kΩ; (j) $R_2 = 229.8$ kΩ; (k) $R_2 = 212.7$ kΩ; (l) $R_2 = 181.8$ kΩ.

6. Random bit generator design and application

6.1. Random bit generator design with the memristive Twin-T oscillator

Chaotic system is used in random bit generator as entropy source because they are complex and very sensitive. The process of quantized random sequence will directly affect the randomness and complexity of the sequence, ultimately affect the security of its application system [Yakcin et al., 2004];[Akogul et al.,...
Various Attractors, Coexisting Attractors, Antimonotonicity in a Simple Four-order Memristive Twin-T Oscillator

Fig. 17. The coexisting attractors in the $x-u$ plane when $R_2 = 416.6$ kΩ, and the scales are $x = 2$ V/div and $u = 5$ V/div for all pictures.

2016]. In this section, The random bit generator design steps are shown in Algorithm 1 as pseudocode.

Algorithm 1 Random bit generator design algorithm pseudocode
1: Start
2: Entering parameters and initial condition of memristive Twin-Tchaotic system
3: Determination of the value of $h$, $n$ and $t_0$ (where $n$ is the length of sequence, $t_0$ is used to bypass the transient stage)
4: Sampling with determination $h$ value
5: for ($i = 1$ to length of all data) do
6: Solving the chaotic system with RK4
7: Convert float to binary number
8: Add periodic perturbation
9: end for
10: The implementation of NIST Tests for bit binary sequency data
11: if test results == pass then
12: Successful results
13: else (test results == false)
14: Return the previous steps and generate bits again
15: end if
16: End

Table 2. Random bit generator NIST-800-22 tests for outputs

<table>
<thead>
<tr>
<th>Statistical tests</th>
<th>$x$ P-value</th>
<th>$z$ P-value</th>
<th>$u$ P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency (monobit) test</td>
<td>0.646952</td>
<td>0.446060</td>
<td>0.166004</td>
</tr>
<tr>
<td>Block frequency test</td>
<td>0.195385</td>
<td>0.759241</td>
<td>0.179463</td>
</tr>
<tr>
<td>Runs test</td>
<td>0.145888</td>
<td>0.375304</td>
<td>0.401250</td>
</tr>
<tr>
<td>Longest-run test</td>
<td>0.981538</td>
<td>0.558705</td>
<td>0.358571</td>
</tr>
<tr>
<td>Binary matrix rank test</td>
<td>0.939268</td>
<td>0.833186</td>
<td>0.444982</td>
</tr>
<tr>
<td>Discrete Fourier transform test</td>
<td>0.699927</td>
<td>0.363621</td>
<td>0.378341</td>
</tr>
<tr>
<td>Nonoverlapping templates test</td>
<td>0.561791</td>
<td>0.844497</td>
<td>0.317707</td>
</tr>
<tr>
<td>Overlapping templates test</td>
<td>0.474914</td>
<td>0.687230</td>
<td>0.454998</td>
</tr>
<tr>
<td>Maurers universal statistical test</td>
<td>0.471521</td>
<td>0.131693</td>
<td>0.906409</td>
</tr>
<tr>
<td>Linear complexity test</td>
<td>0.625404</td>
<td>0.115927</td>
<td>0.261935</td>
</tr>
<tr>
<td>Serial test 1</td>
<td>0.544026</td>
<td>0.665659</td>
<td>0.147126</td>
</tr>
<tr>
<td>Serial test 2</td>
<td>0.298254</td>
<td>0.696731</td>
<td>0.340348</td>
</tr>
<tr>
<td>Approximate entropy test</td>
<td>0.517227</td>
<td>0.704865</td>
<td>0.063250</td>
</tr>
<tr>
<td>Cumulative sums test [left]</td>
<td>0.974737</td>
<td>0.294050</td>
<td>0.098828</td>
</tr>
<tr>
<td>Cumulative sums test [right]</td>
<td>0.940276</td>
<td>0.534794</td>
<td>0.190234</td>
</tr>
<tr>
<td>Random excursions test $(x = 4)$</td>
<td>0.515355</td>
<td>0.557664</td>
<td>0.072948</td>
</tr>
<tr>
<td>Random excursions variant $(x = 9)$</td>
<td>0.332342</td>
<td>0.296133</td>
<td>0.178385</td>
</tr>
</tbody>
</table>
We set $h = 0.02$, $t_0 = 500$, and the system parameters $(a = 7.5, b = 0.35, c = 3.5, m = 10, n = 1, k = 0.1, y = 0.1)$, and the initial conditions are $[1, 0, 0, 0]$. After all these processes based on above algorithm, the NIST-800-22 statistical tests are used for success of the random bit generator design. The NIST-800-22 tests consist of 16 different tests. The results of the NIST-800-22 tests should be greater than 0.01 for success. Results of NIST-800-22 test for 1 Mbit number series generated with chaotic system are given in Table 2. In the table 2, values are given when $x$ or $y$ or $z$ is $-4$ for Random Excursions test, and $x$ or $y$ or $z$ is $-9$ Random Excursions Variant test. Random Excursion and Random Excursions Variant have successful results for the outputs $(x, y, z, w)$, however, $y$ output does not pass all the tests under different pattern-matching of Nonoverlapping templates test. From the Table 2, it is found that the outputs of $x$, $z$ and $w$ are all successfully pass all the tests (all P-values are larger than or equal to 0.01). Fig. 18 shows three output serials (We only select partial data to observe clearly). The Twin-T memristive oscillator can be considered as multi-channel random bit generator and be used in other engineering and information technology applications that require randomness.

![Figure 18](image_url)  
Fig. 18. Three output serials: outputs $x$ (red), outputs $y$ (blue), outputs $u$ (green).

### 6.2. Image Encryption Application

In this section, a image encryption with random bit generator is realized as an example of chaos-based applications. We select gray scale image Lena ($256 \times 256$) as example. The steps of the encryption and decryption process are shown in Algorithm 2.

Algorithm 2 Chaos-based image encryption and decryption algorithm pseudo code

1: Start
2: Getting original image $P (M \times N)$ to be encrypted
3: Original image data convert to one dimension vector $A$
4: Getting ready tested random bit sequence $B (8 \times M \times N )$ for keys
5: Convert bit sequence $B$ to 8-bit binary sequence $C$ (The length is $M \times N$
6: for $i = 1$ to all original image data do
7: 8-bit binary sequence $C$ bit-level xor original image data (one dimension vector $A$)
8: end
9: Encrypted image $D$
10: Encrypted image data convert to one dimension vector $B$
11: for $i = 1$ to all encrypted image data do
12: 8-bit binary sequence $C$ bit-level xor encrypted image data (one dimension vector $B$)
13: end
Fig. 19. Applicability of the encryption and decryption using the proposed random bit generator: (a) original image; (b) encrypted image; (c) decrypted image.

Fig. 20. The histogram of different images: (a) original image; (b) encrypted image with tested random bit sequence $x$; (c) encrypted image with tested random bit sequence $y$; (d) encrypted image with tested random bit sequence $z$.

Decrypted image
End

As shown in Algorithm 2, XOR operator is used for the encryption process. The original image, encrypted image and decrypted image are described in Fig. 18. From Fig. 19 (a) and Fig. 19 (c), there are no distortion between the original and decrypted data. The histogram values are seen in Fig. 20. Obviously, the results in Fig. 20 reveal that the proposed encrypted images that are uniformly distributed as noted in Figs 20 (b) - (d). This concludes that the proposed encryption algorithm is robust against statistical attacks, and encryptions with different tested random bit sequence were performed successfully.

7. Conclusion

In this paper, we present a simple four-order memristive Twin-T oscillator. This circuit was obtained by replacing the resistor in a Twin-T oscillator with generalized flux-controlled memristor. The complex
dynamical behaviors are proven via nonlinear dynamics tools, simulations in Matlab, and experiments. Especially, by varying one parameter, various periodic orbits and various chaotic attractors can be obtained simple four-order memristive Twin-T oscillator. At the same time, we observed the phenomena of anti-monotonicity, coexisting bifurcation modes. Random bit generator for signal application is proposed, and numerical results illustrate the usefulness of the random bit generator. Furthermore, image encryptions are carried out with the random bit sequence designed here. The proposed circuit with inductor-free is suitable for IC design and may be helpful for chaos-based communication and electronic measurement systems.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (No. 61571185), The Natural Science Foundation of Human Province, China(No. 2016J12030), the Open Fund Project of Key Laboratory in Human Universities(No. 15K027), Scientific Research Fund of Human University of science and engineering (No.17XKY070), and Scientific Research Fund of Human Provincial Education Department (No.12A0654, No.16C0683).

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