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Time-controllable combinatorial inner synchronization and outer synchronization of anti-star networks and its application in secure communication

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ABSTRACT

In this paper, a time-controllable combinatorial inner synchronization and outer synchronization of anti-star networks, each of which consists of four-wing hyper-chaotic system as node dynamics, is investigated. Based on the adaptive technique and the stability of Lyapunov function, some sufficient conditions, which can ensure the realization of not only combinatorial inner synchronization within an anti-star network with unknown parameters and external perturbations in the computable time, but also combinatorial outer synchronization between different sub-networks with external perturbations in the computable time, are obtained. Moreover, a simple secure communication scheme, which is based on the adaptive combinatorial outer synchronization between different sub-networks under the influence of stochastic noise and time-delay, is presented. Numerical simulation results show the feasibility and validity of the proposed method.

1. Introduction

Chaos is a very interesting nonlinear phenomenon and it has been widely studied in the past three decades. Since the concept for constructing synchronization of coupled chaotic systems was proposed by Pecora and Carroll in 1990 [1], the control and synchronization problems of chaotic systems have been intensively investigated due to their potential applications in various fields such as in secure communication, chemical reactions, biological systems and many other fields. Up to now, various types of control method such as active control [2], adaptive control [3], sliding mode control [4], back-stepping control [5], linear and nonlinear feedback control [6,7], impulse control [8], pinning control [9], etc. have been successfully used in the complete synchronization [10], phase synchronization [11], lag synchronization [12], generalized synchronization [13], projective synchronization [14], modified projective synchronization [15], Q–S synchronization [16], novel compound synchronization [17] and so on. However, most of the aforementioned works have focused on the synchronization of the one-to-one system, which limits the application range of synchronization in the reality to some extent. As the fact that the complex dynamical networks including neural networks, power grids, food webs, ecosystems, the World Wide Webs, etc. are ubiquitous in our daily lives, it seems that the study of the dynamical structure and the synchronization of complex networks may have great value to understand the functions of the real-world. In this regard, at present, more and more researchers begin to draw their attention to the synchronization of large and complex network with multiple nodes. Considering the complexity of the network structure, the synchronization of networks is still an open and challenging problem. In

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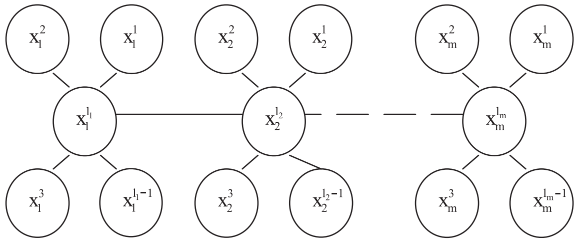

Fig. 1. The network structure diagram with m anti-star sub-networks.

general, the synchronization of networks can be roughly divided into two kinds: inner synchronization and outer synchronization. In the past decade, the majority of works in synchronization of the network has focused on the inner synchronization, which is concerned with the synchronization among the nodes within a network [18–20]. While in the real world, there are a variety of complex networks with the same or different topological structure, most of them need to realize the synchronization and control between different networks namely outer synchronization. Recently, Wu et al. have investigated the outer synchronization between drive-response networks with non-identical topological structure and unknown parameters [21]. In [22], the outer synchronization of uncertain complex delayed networks with adaptive coupling method has been studied. The generalized outer synchronization between two different delay-coupled complex dynamical networks with noise perturbation has been investigated in [23]. Moreover, the finite-time control techniques which have been demonstrated with better disturbance rejection and robustness against uncertainties [24], have become a research hotspot for its practical application value in engineering. In [25], He et al. have investigated the problem of finite-time mixed outer synchronization of complex networks with coupling time-varying delay, some novel stability criteria for the synchronization between drive and response complex networks with coupling time-varying delay are derived by using the Lyapunov stability theory and linear matrix inequalities. Sun et al. have proposed a finite-time stochastic outer synchronization between two different complex dynamical networks with noise perturbation based on the finite-time stability theory of stochastic differential equations [26]. In [27], a finite-time synchronization between two complex networks with non-delayed and delayed coupling has been proposed by using the impulsive control and the periodically intermittent control. However, most of the above works on the outer synchronization are limited to only two networks, and there are some difficulties to extend these methods to the synchronization between the three or more ones. As in the reality, there are still three or more sub-networks that need to synchronize with each other, such as in the secure communication, to improve the security level of the transmitted signal, the transmitted signal may be split into several parts, each part is loaded in different sub-networks respectively, these sub-networks need to achieve the recovery of the information signal in a synchronous way. Therefore, it has become more and more important and meaningful if we can put forward a more general method to deal with the synchronization of multiple sub-networks.

On the other hand, due to the fact that the chaotic system offers some advantages in communication systems such as broadband noise-like waveform, prediction difficulty, etc., the synchronization of chaotic systems as an effective encryption mechanism has been widely used in the secure communication, and the level of security is mostly dependent on the complexity level of the drive's dynamics and the formation of the driving signal as well as the modulation scheme used [28]. In the traditional chaotic secure communication scheme, the information signal to be transmitted is added to only one chaotic system, which has proved to be obtained easily by the attackers [29]. In order to improve the security level of the transmitter signal, similar to the serial packet transport, we can split the original signal into several parts, each part with different weighting is loaded in different complex dynamical systems with very complex dynamic behavior, such as hyper-chaotic systems. Knowing the weightings as secret keys, the receiver will recover the information signal accurately if all parts from different sub-networks arrive at the destination in the synchronous way. It is qualitative to say that the method may have much stronger anti-attack and anti-translated capability than the traditional transmission mode [28]. Furthermore, due to the finite information transmission and processing speed among the network nodes, the transmitted signal is still inevitably influenced by all kinds of random factors such as channel noise, time-delay, etc. Effectively reflecting these stochastic factors can help us recognize the real world more reasonably, therefore, it has more practical value to study the chaos synchronization of stochastic system.

In addition, how to realize the synchronization of large scale complex networks with only fewer controllers is extremely challenging and far-reaching significance. The star topological structure which is known for its simple structure has been studied in the synchronization of complex network [30–32], but it needs too many controllers in realizing the inner synchronization and the outer synchronization of complex networks. How to reduce the number of controllers for the synchronization of complex networks seems particularly important. The pinning control technique can yet be regarded as a kind of good way to decrease the number of controllers. Thus, a natural question may arise: can we propose a simple topological structure by making full use of the superiority of the star topological structure and pinning control scheme to reduce the number of controllers for the synchronization of large scale complex networks? If this is possible, the method will realize the synchronization of the nodes within a network or from different ones with only fewer controllers. We may as well name it anti-star topological structure, which is similar to the star topological structure proposed in [30,31]. The difference between them is: the star topological structure needs multiple controllers to realize the synchronization because it lets the peripheral nodes as the response systems and the central node as the drive system in the divergent way, while the anti-star topological structure performs in the opposite way, it achieves the combinatorial synchronization between multiple peripheral nodes and a central node in the convergence way and needs only one controller, which can significantly reduce costs in realizing the synchronization of complex networks.

Motivated by all of the above discussion, in this paper, we aimed to realize the combinatorial inner synchronization within an anti-star network and the combinatorial outer synchronization between different sub-networks in the computable time. By introducing a switch control scheme to the central node, we can choose different connection matrices to couple central node and peripheral nodes within a network and all the central nodes in different sub-networks, respectively. The proposed method can realize not only the combinatorial inner synchronization within an anti-star network with unknown parameters and external disturbances but also the combinatorial outer synchronization between different sub-networks with external disturbances in the computable time. Finally, we set the application on the recovery of the transmitted signal under the influence of stochastic noise and time-delay as an example to demonstrate the feasibility and validity of the proposed method.

The organization of the paper is listed as follows. A network modeling and some preliminaries are given in Section 2. In Section 3, some main results for the combinatorial inner synchronization within a network and the combinatorial outer synchronization between different sub-networks are proposed. Two numerical simulation results are given in Section 4. In Section 5, a scheme of secure communication based on the adaptive combinatorial outer synchronization between different sub-networks with stochastic perturbation and time-delay is presented. The conclusions are finally drawn in Section 6.

2. Network modeling and preliminaries

Based on the concept of birds of a feather flock together, the nodes in the same network always have the same system model. We may assume that the complex network consists of n nodes with m anti-star sub-networks which are described as $V_1, V_2, \ldots, V_m$, and every sub-network has l_i $(i = 1, 2, \ldots, m)$ nodes, which is to say that we have $\sum_{k=1}^{m} l_k = n$. Let $\phi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$, if node i belongs to the jth sub-network, then we have $\phi_i = j$. The nodes which belong to the same sub-network are connected by anti-star topological structure, the central node is selected as the response system and all the peripheral nodes are selected as the drive systems, the network structure diagram with m anti-star sub-networks is described as Fig. 1. In the Fig. 1, x_k^i $(k = 1, 2, \ldots, m;\ i = 1, 2, \ldots, l_k)$ represents a node in a sub-network, the subscript k represents the kth sub-network, the superscript i represents the ith node. The nodes in their subordinate sub-network are marked as $1, 2, \ldots, l_k$ from outside to inside in a counter-clockwise direction, and the central node is marked as l_k. The line from x_k^i to $x_k^{l_k}$ represents a connection between node i and node l_k in the kth sub-network, and there are still some connections between different sub-networks, which is omitted in this diagram. The corresponding mathematical expression can be written as follows:

$$\dot{x}^i = f_{\phi_i}(x^i) + F_{\phi_i}(x^i)\theta_i + \Delta f_i(x^i) + \sum_{j=1}^{n} c_{i,j} x^j, \quad i = 1, 2, \ldots, n \tag{1}$$

where the superscript i represents the ith node, $x^i = [x_1^i,\ x_2^i,\ \ldots,\ x_n^i]^T \in R^n$ represents the state vector of node i, $f_{\phi_i}(x^i) = [f_{\phi_i}(x_1{}^i),\ f_{\phi_i}(x_2{}^i),\ \ldots,\ f_{\phi_i}(x_n{}^i)]^T \in R^n$ is the continuous nonlinear vector function, $F_{\phi_i}(x^i)$ is a $n \times m_0$ matrix, $\theta_i \in R^{m_0 \times 1}$ represents a unknown vector parameter of node i, and $\Delta f_i(x^i) \in R^{n \times 1}$ denotes the vector of external disturbance. $C = (c_{i,j})_{n \times n}$ denotes a connection matrix, in which $c_{i,j} \neq 0$ if there is a connection from node i to node j $(i \neq j)$, and satisfies the condition that the sum of all the elements is equal to 0 in each line. For the combinatorial inner synchronization within a network, the aim is to design a suitable controller on the response system to realize the synchronization of multiple drive systems and only one response system, as the fact that the nodes in each sub-network can perform well alone, we do not consider the connections between different sub-networks at this stage, so the drive systems in the kth $(k = 1, 2, \ldots, m)$ anti-star sub-network can be corrected as

$$\dot{x}_k^i = f_k(x_k{}^i) + F_k(x_k{}^i)\theta_i + \Delta f_i(x_k{}^i) + \sum_{j=1}^{l_k} c_{i,j} x_k^j, \quad i = 1, 2, \ldots, l_k - 1 \tag{2}$$

The corresponding response system can be described as

$$\dot{x}_k^{l_k} = f_k(x_k{}^{l_k}) + F_k(x_k{}^{l_k})\theta_{l_k} + \Delta f_{l_k}(x_k{}^{l_k}) + \sum_{j=1}^{l_k} c_{l_k,j} x_k^j + u_k(t) \tag{3}$$

where $u_k(t) \in R^{n \times 1}$ is an appropriate controller to be designed.

Definition 1. If there are some constant diagonal matrices A_i $(i = 1, 2, \ldots, l_k)$ with suitable dimension, and A_{l_k} is a reversible matrix such that $\lim_{t\to\infty} \left\| \sum_{i=1}^{l_k-1} A_i x_k^i - A_{l_k} x_k^{l_k} \right\| = 0$, then the drive systems (2) and the response system (3) are called to realize the combinatorial inner synchronization within the kth $(k = 1, 2, \ldots, m)$ anti-star sub-network, where $\|\bullet\|$ represents the matrix norm.

By setting the error state as $e_k(t) = \sum_{i=1}^{l_k-1} A_i x_k^i - A_{l_k} x_k^{l_k}$, where $A_1, A_2, \ldots, A_{l_k}$ are diagonal matrices with suitable dimension, according to the driver systems (2) and response system (3), we can get the combinatorial inner synchronization error system as

$$\dot{e}_k(t) = \sum_{i=1}^{l_k-1} A_i f_k({x_k}^i) - A_{l_k} f_k({x_k}^{l_k}) + \sum_{i=1}^{l_k-1} A_i F_k({x_k}^i)\theta_i - A_{l_k} F_k({x_k}^{l_k})\theta_{l_k} + \sum_{i=1}^{l_k-1} A_i \Delta f_i({x_k}^i) - A_{l_k} \Delta f_{l_k}({x_k}^{l_k}) + \sum_{j=1}^{l_k} \left(\sum_{i=1}^{l_k-1} A_i c_{i,j} x_k^j - A_{l_k} c_{l_k,j} x_k^j \right) - A_{l_k} u_k(t) \tag{4}$$

After the nodes within a sub-network have synchronized, all the unknown parameters of the nodes have been identified exactly, then we can realize the combinatorial outer synchronization between different sub-networks with exact system model, so the drive and response systems can be described as

$$\dot{\tilde{x}}^{l_k} = f_k(\tilde{x}^{l_k}) + \Delta f_{l_k}(\tilde{x}^{l_k}) + \sum_{j=1}^{m} \tilde{c}_{l_k,j} \tilde{x}^{l_j}, \quad k = 1, 2, \ldots, m-1 \tag{5}$$

$$\dot{\tilde{x}}^{l_m} = f_m(\tilde{x}^{l_m}) + \Delta f_{l_m}(\tilde{x}^{l_m}) + \sum_{j=1}^{m} \tilde{c}_{l_m,j} \tilde{x}^{l_j} + \tilde{u}(t) \tag{6}$$

where $\tilde{x}^{l_k} = [\tilde{x}_1^{l_k}, \tilde{x}_2^{l_k}, \ldots, \tilde{x}_n^{l_k}]^T \in R^n$ denotes the state vector of node l_k, $f_k(\tilde{x}^{l_k}) = [f_k(\tilde{x}_1^{l_k}), f_k(\tilde{x}_2^{l_k}), \ldots, f_k(\tilde{x}_n^{l_k})]^T$ is the continuous nonlinear vector function, and $\Delta f_{l_k}(\tilde{x}^{l_k}) \in R^{n\times 1}$ represents the vector of external disturbance of node l_k. $\tilde{C} \in R^{m\times m}$ is a connection matrix, in which $\tilde{c}_{i,j}$ represents the connection weight between the ith central node and the jth central node. $\tilde{u}(t) \in R^{n\times 1}$ is a controller to be designed.

Definition 2. If there are some constant diagonal matrices $\tilde{A}_k$ $(k = 1, 2, \ldots, m)$ with suitable dimension, and $\tilde{A}_m$ is a reversible matrix such that $\lim_{t\to\infty} \left\| \sum_{i=1}^{m-1} \tilde{A}_i \tilde{x}^{l_i} - \tilde{A}_m \tilde{x}^{l_m} \right\| = 0$, then the drive systems (5) and response system (6) are called to realize the combinatorial outer synchronization between different sub-networks, where $\|\bullet\|$ represents the matrix norm.

Let $\tilde{e}(t) = \sum_{i=1}^{m-1} \tilde{A}_i \tilde{x}^{l_i} - \tilde{A}_m \tilde{x}^{l_m}$ as the error state, and $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_m$ are diagonal matrices with suitable dimension. Then the combinatorial outer synchronization error system can be described as

$$\dot{\tilde{e}}(t) = \sum_{i=1}^{m-1} \tilde{A}_i f_i(\tilde{x}^{l_i}) - \tilde{A}_m f_m(\tilde{x}^{l_m}) + \sum_{i=1}^{m-1} \tilde{A}_i \Delta f_{l_i}(\tilde{x}^{l_i}) - \tilde{A}_m \Delta f_{l_m}(\tilde{x}^{l_m}) + \sum_{j=1}^{m} \left(\sum_{i=1}^{m-1} \tilde{A}_i \tilde{c}_{l_i,j} \tilde{x}^{l_j} - \tilde{A}_m \tilde{c}_{l_m,j} \tilde{x}^{l_j} \right) - \tilde{A}_m \tilde{u}(t) \tag{7}$$

Prior to designing the synchronizing controller in the network, some assumptions must be noted as follows:

A1. The uncertain parameters θ_i are all norm bounded, such as $\|\theta_i\| \leqslant \delta_{\theta_i}$, where δ_{θ_i} are known positive constants and $i = 1, 2, \ldots, l_k;\ k = 1, 2, \ldots, m$.

A2. The unknown uncertainties $\Delta f_i(x^i)$ $(i = 1, 2, \ldots, l_k)$ are all bounded, which means that there are some positive constants ρ_k, ρ, such that $\left\| \sum_{i=1}^{l_k-1} A_i \Delta f_i(x^i) - A_{l_k} \Delta f_{l_k}(x^{l_k}) \right\| \leqslant \rho_k,\ k = 1, 2, \ldots, m$ for the combinatorial inner synchronization and $\left\| \sum_{i=1}^{m-1} \tilde{A}_i \Delta f_{l_i}(\tilde{x}^{l_i}) - \tilde{A}_m \Delta f_{l_m}(\tilde{x}^{l_m}) \right\| \leqslant \rho$ for the combinatorial outer synchronization.

A3. There is a sufficient small positive constant ε, such that $\left\| \theta_i - \hat{\theta}_i \right\| \geqslant \varepsilon$ $(i = 1, 2, \ldots, l_k;\ k = 1, 2, \ldots, m)$. Note that $\left\| \theta_i - \hat{\theta}_i \right\| \geqslant 0$, and ε is a presupposed positive constant that can be chosen arbitrarily small.

Remark 1. The purpose of introducing ε is to avoid the unknown parameters from appearing in controllers and parameters update laws.

Now we give the definition of the time-controllable combinatorial synchronization between the drive systems (2) (or (5)) and the response system (3) (or (6)), and some lemmas which will be used later.

Definition 3. The drive systems (2) (or (5)) and the response system (3) (or (6)) are defined to be combinatorial synchronized in a network (or between different sub-networks) with the computable time if, for a suitable feedback controller, there is a constant $T > 0$, such that $\lim_{t\to T} \|e\| = 0$ and $\|e\| = 0$ for $t > T$.

Lemma 1. *Assume that a continuous and positive definite function $V(t)$ satisfies the following differential inequality [33]:*

$$\dot{V}(t) \leqslant -cV^{\eta}(t)$$

for any $t > 0, V(t_0) \geqslant 0$, where $c > 0, 0 < \eta < 1$ are all constants. Then, for any given $t_0, V(t)$ satisfies the following inequality:

$$V^{1-\eta}(t) \leqslant V^{1-\eta}(t_0) - c(1-\eta)(t-t_0), \quad t_0 \leqslant t \leqslant T$$

and $V(t) = 0$ for $t > T$ with $T = t_0 + \frac{V^{1-\eta}(t_0)}{c(1-\eta)}$.

Lemma 2. *For any real number a_i $(i = 1, 2, \ldots, n)$ and $0 < c \leqslant 1$, the following inequality[34] holds:*

$$(|a_1| + |a_2| + \cdots + |a_n|)^c \leqslant |a_1|^c + |a_2|^c + \cdots + |a_n|^c$$

Lemma 3 (*Barbalat lemma [35]*). *If $w : R_+ \to R_+$ is a uniformly continuous function for $t \geqslant 0$ and if the limit of the integral $\lim\limits_{t\to\infty} \int_0^t w(\lambda)d\lambda$ exists and is finite, then $\lim\limits_{t\to\infty} w(t) = 0$.*

3. Main results for the combinatorial inner synchronization and outer synchronization

In this section, we shall establish some synchronization criteria for the combinatorial inner synchronization within a network and the combinatorial outer synchronization between different sub-networks, respectively.

3.1. The combinatorial inner synchronization criterion within the kth sub-network

Theorem 1. *Let (**A1**)–(**A3**) hold and the controller $u_k(t)$ is given by*

$$u_k(t) = A_{l_k}^{-1}\left(\sum_{i=1}^{l_k-1} A_i f_k(x_k{}^i) - A_{l_k} f_k(x_k{}^{l_k}) + \sum_{i=1}^{l_k-1} A_i F_k(x_k{}^i)\hat{\theta}_i - A_{l_k} F_k(x_k{}^{l_k})\hat{\theta}_{l_k} + \rho_k sign(e_k) + \frac{e_k}{\|e_k\|} + \sum_{j=1}^{l_k}\left(\sum_{i=1}^{l_k-1} A_i c_{i,j} x_k^j - A_{l_k} c_{l_k,j} x_k^j\right) + 2\sum_{i=1}^{l_k} \frac{\delta_{\theta_i}^2 + \delta_{\theta_i}\left\|\hat{\theta}_i\right\|}{\varepsilon} \frac{e_k}{\|e_k\|^2}\right) \tag{8}$$

where $sign(e_k)$ $(k = 1, 2, \ldots, m)$ stands for the sign function. Meanwhile, the update laws of unknown parameters $\hat{\theta}_i$ $(i = 1, 2, \ldots, l_k)$ in the kth sub-network are taken as follows:

$$\begin{aligned} \dot{\hat{\theta}}_i &= F_k^T(x_k{}^i)A_i^T e_k + \frac{\Delta\theta_i - \hat{\theta}_i}{\varepsilon}, \quad i = 1, 2, \ldots, l_k - 1 \\ \dot{\hat{\theta}}_{l_k} &= -F_k^T(x_k{}^{l_k})A_{l_k}^T e_k + \frac{\Delta\theta_{l_k} - \hat{\theta}_{l_k}}{\varepsilon}. \end{aligned} \tag{9}$$

where $\Delta\theta_i = [\delta_{\theta_{i1}}, \delta_{\theta_{i2}}, \ldots \delta_{\theta_{im_0}}]^T \in R^{m_0\times 1}$, $\delta_{\theta_{ij}}$ is the upper bound of the jth component of the unknown parameters θ_i, and δ_{θ_i} satisfies the condition that $\delta_{\theta_i} \geqslant \sqrt{\sum_{j=1}^{m_0} \delta_{\theta_{ij}}^2} (i = 1, 2, 3 \ldots, l_k)$. Then, the combinatorial inner synchronization within the kth sub-network can be achieved globally asymptotically in the computable time.

The proof of the Theorem 1 is standard but lengthy, we shall present it in Appendix A of this paper later.

Remark 2. The constant ρ_k can be estimated by simulation, and the value of the designed constant ρ_k must be large enough.

Remark 3. The magnitude of $e_k/\|e_k\|$ in the controller $u_k(t)$ will turn to infinity as $e_k \to 0$, in order to avoid the occurrence of this phenomenon, we can add a sufficient small positive constant $\tilde{\varepsilon}$ to its denominator in practice.

3.2. The combinatorial outer synchronization criterion between different sub-networks

After all of the systems parameters have been identified by the combinatorial inner synchronization within a sub-network, the central nodes in different sub-networks have precise system models, then we can build the combinatorial outer synchronization criterion between different sub-networks.

Theorem 2. *Let (**A2**) holds and the controller $\tilde{u}(t)$ is given by*

$$\tilde{u}(t) = \widetilde{A}_m^{-1}\left(\sum_{i=1}^{m-1} \widetilde{A}_i f_i(\tilde{x}^{l_i}) - \widetilde{A}_m f_m(\tilde{x}^{l_m}) + \rho sign(\tilde{e}) + \frac{\tilde{e}}{\|\tilde{e}\|} + \sum_{j=1}^{m}\left(\sum_{i=1}^{m-1} \widetilde{A}_i \tilde{c}_{l_i,j} \tilde{x}^{l_j} - \widetilde{A}_m \tilde{c}_{l_m,j} \tilde{x}^{l_j}\right)\right) \tag{10}$$

Then, the combinatorial outer synchronization between different sub-networks can be achieved globally asymptotically in the computable time.

The detailed proof of Theorem 2 is presented in Appendix B.

4. Numerical simulations for the combinatorial inner synchronization and outer synchronization

In the following section, the four-wing hyper-chaotic system [36] which is proposed by Li et al. is taken as the node dynamics to validate the feasibility and effectiveness of the proposed approach. The dynamic equation of the system can be given as follows:

$$\begin{cases} \dot{x} = ax - yz + kw \\ \dot{y} = -by + xz \\ \dot{z} = -cz + xy + dx \\ \dot{w} = -mx \end{cases} \tag{11}$$

where a, b, c, d, k, m are the system parameters, $x, y, z, w \in R$ are the state variables. When the parameters are chosen as $a = 4, b = 12, c = 5.5, d = 1, k = 2.5, m = 1$, the system (11) is hyper-chaotic with Lyapunov exponents $\lambda_1 = 0.7052$, $\lambda_2 = 0.1106, \lambda_3 = 0, \lambda_4 = -14.2811$, and the Lyapunov dimension is $d_L = 3.0567$. The phase diagram of the four-wing hyper-chaotic attractor is shown in Fig. 2.

Considering the complexity of large scale network, we just assume $l_k = 4$ $(k = 1, 2, 3, 4)$ in the following simulation, which is to say that the network is composed of four anti-star sub-networks and every sub-network has three peripheral nodes and only one central node. The four-wing hyper-chaotic system is selected as the node dynamics. Star-like connection matrix and chain connection matrix are chosen as the interior connection matrix and the exterior connection matrix respectively. Numerical simulation results for the combinatorial inner synchronization and outer synchronization show the feasibility and validity of the proposed scheme. The concrete network structure diagram is described as Fig. 3.

4.1. Combinatorial inner synchronization within an anti-star sub-network

For the combinatorial inner synchronization within an anti-star sub-network, we label the nodes from outside to inside, so the star-like connection matrix can be chosen as

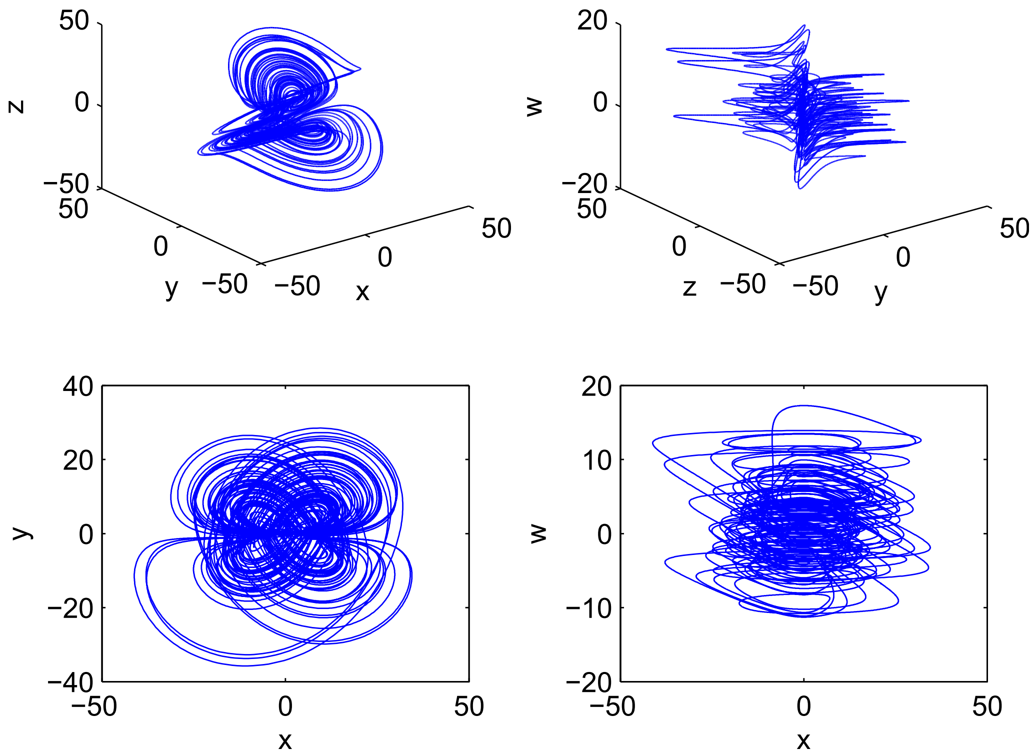

Fig. 2. The phase diagram of the four-wing hyper-chaotic system.

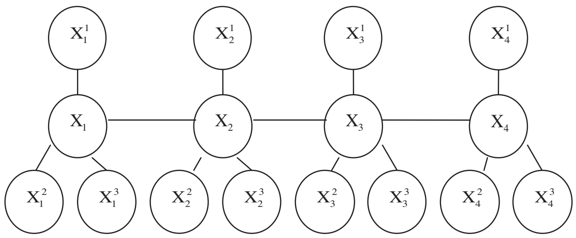

Fig. 3. The network structure diagram formed by four sub-networks with anti-star topological structure.

$$C = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \tag{12}$$

Both the drive systems and the response system can be expressed as

$$\begin{aligned}
\dot{x}^1 &= \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \\ \dot{w}_1 \end{pmatrix} = \underbrace{\begin{pmatrix} ax_1 - y_1 z_1 \\ -by_1 + x_1 z_1 \\ -cz_1 + x_1 y_1 \\ -mx_1 \end{pmatrix}}_{f_1(x^1)} + \underbrace{\begin{pmatrix} w_1 & 0 \\ 0 & 0 \\ 0 & x_1 \\ 0 & 0 \end{pmatrix}}_{F_1(x^1)} \underbrace{\begin{pmatrix} k_1 \\ d_1 \end{pmatrix}}_{\theta_1} + \underbrace{\begin{pmatrix} 0.1\sin(x_1) \\ 0.2\sin(y_1) \\ 0.3\sin(z_1) \\ 0.4\sin(w_1) \end{pmatrix}}_{\Delta f_1(x^1)} + \underbrace{\begin{pmatrix} x_1 - x_4 \\ y_1 - y_4 \\ z_1 - z_4 \\ w_1 - w_4 \end{pmatrix}}_{\Sigma c_{1,j} x^j} \\
\dot{x}^2 &= \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \\ \dot{z}_2 \\ \dot{w}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} ax_2 - y_2 z_2 \\ -by_2 + x_2 z_2 \\ -cz_2 + x_2 y_2 \\ -mx_2 \end{pmatrix}}_{f_2(x^2)} + \underbrace{\begin{pmatrix} w_2 & 0 \\ 0 & 0 \\ 0 & x_2 \\ 0 & 0 \end{pmatrix}}_{F_2(x^2)} \underbrace{\begin{pmatrix} k_2 \\ d_2 \end{pmatrix}}_{\theta_2} + \underbrace{\begin{pmatrix} -0.1\cos(x_2) \\ -0.2\cos(y_2) \\ -0.3\cos(z_2) \\ -0.4\cos(w_2) \end{pmatrix}}_{\Delta f_2(x^2)} + \underbrace{\begin{pmatrix} x_2 - x_4 \\ y_2 - y_4 \\ z_2 - z_4 \\ w_2 - w_4 \end{pmatrix}}_{\Sigma c_{2,j} x^j} \\
\dot{x}^3 &= \begin{pmatrix} \dot{x}_3 \\ \dot{y}_3 \\ \dot{z}_3 \\ \dot{w}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} ax_3 - y_3 z_3 \\ -by_3 + x_3 z_3 \\ -cz_3 + x_3 y_3 \\ -mx_3 \end{pmatrix}}_{f_3(x^3)} + \underbrace{\begin{pmatrix} w_3 & 0 \\ 0 & 0 \\ 0 & x_3 \\ 0 & 0 \end{pmatrix}}_{F_3(x^3)} \underbrace{\begin{pmatrix} k_3 \\ d_3 \end{pmatrix}}_{\theta_3} + \underbrace{\begin{pmatrix} 0.1\sin(x_3) \\ -0.2\cos(y_3) \\ 0.3\cos(z_3) \\ -0.4\sin(w_3) \end{pmatrix}}_{\Delta f_3(x^3)} + \underbrace{\begin{pmatrix} x_3 - x_4 \\ y_3 - y_4 \\ z_3 - z_4 \\ w_3 - w_4 \end{pmatrix}}_{\Sigma c_{3,j} x^j}
\end{aligned} \tag{13}$$

$$\begin{aligned}
\dot{x}^4 &= \begin{pmatrix} \dot{x}_4 \\ \dot{y}_4 \\ \dot{z}_4 \\ \dot{w}_4 \end{pmatrix} = \underbrace{\begin{pmatrix} ax_4 - y_4 z_4 \\ -by_4 + x_4 z_4 \\ -cz_4 + x_4 y_4 \\ -mx_4 \end{pmatrix}}_{f_4(x^4)} + \underbrace{\begin{pmatrix} w_4 & 0 \\ 0 & 0 \\ 0 & x_4 \\ 0 & 0 \end{pmatrix}}_{F_4(x^4)} \underbrace{\begin{pmatrix} k_4 \\ d_4 \end{pmatrix}}_{\theta_4} + \underbrace{\begin{pmatrix} -0.1\cos(x_4) \\ 0.2\sin(y_4) \\ 0.3\cos(z_4) \\ -0.4\sin(w_4) \end{pmatrix}}_{\Delta f_4(x^4)} \\
&+ \underbrace{\begin{pmatrix} -x_1 - x_2 - x_3 + 3x_4 \\ -y_1 - y_2 - y_3 + 3y_4 \\ -z_1 - z_2 - z_3 + 3z_4 \\ -w_1 - w_2 - w_3 + 3w_4 \end{pmatrix}}_{\Sigma c_{4,j} x^j} + \underbrace{\begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{pmatrix}}_{u(t)}
\end{aligned} \tag{14}$$

where the superscript i $(i = 1, 2, 3, 4)$ represents the ith node, the parameters are selected as $a = 4, b = 12, c = 5.5, m = 1$, and k_i, d_i $(i = 1, 2, 3, 4)$ are unknown. Let the error state as

$$e = A_1 x^1 + A_2 x^2 + A_3 x^3 - A_4 x^4 = \begin{pmatrix} \alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3 - m_1 x_4 \\ \alpha_2 y_1 + \beta_2 y_2 + \gamma_2 y_3 - m_2 y_4 \\ \alpha_3 z_1 + \beta_3 z_2 + \gamma_3 z_3 - m_3 z_4 \\ \alpha_4 w_1 + \beta_4 w_2 + \gamma_4 w_3 - m_4 w_4 \end{pmatrix} \tag{15}$$

where $A_1 = diag(\alpha_1, \alpha_2, \alpha_3, \alpha_4), A_2 = diag(\beta_1, \beta_2, \beta_3, \beta_4), A_3 = diag(\gamma_1, \gamma_2, \gamma_3, \gamma_4), A_4 = diag(m_1, m_2, m_3, m_4)$, and according to the detailed theory analysis presented in Section 3, the combinatorial inner synchronization within an anti-star sub-network is easily realized. In the simulation process, we take the matrices as $A_1 = diag(3, 2, 1, 4), A_2 = diag(2, 3, 3, 2)$,

$A_3 = diag(3,2,4,3), A_4 = diag(2,3,2,1)$. The initial conditions for the drive systems and the response system are chosen as $(x_{10}, y_{10}, z_{10}, w_{10}) = (4,-5,12,-8), (x_{20}, y_{20}, z_{20}, w_{20}) = (6,8,-2,10), (x_{30}, y_{30}, z_{30}, w_{30}) = (5,-4,8,16)$ and $(x_{40}, y_{40}, z_{40}, w_{40}) = (2,6,10,-4)$. Thus, we can obtain $(e_{10}, e_{20}, e_{30}, e_{40}) = (35,-12,18,40)$. The initial values of estimated parameters are chosen as $(\hat{k}_1, \hat{d}_1) = (2.05, 0.05), (\hat{k}_2, \hat{d}_2) = (1.25, 0.45), (\hat{k}_3, \hat{d}_3) = (2.15, 0.25)$ and $(\hat{k}_4, \hat{d}_4) = (2.05, 0.35)$. The upper bound of the *jth* $(j = 1,2)$ component of the unknown parameters θ_i $(i = 1,2,3,4)$ are selected as $(\delta_{\theta_{11}}, \delta_{\theta_{12}}) = (2.55, 1.05), (\delta_{\theta_{21}}, \delta_{\theta_{22}}) = (2.6, 1.1), (\delta_{\theta_{31}}, \delta_{\theta_{32}}) = (2.65, 1.15), (\delta_{\theta_{41}}, \delta_{\theta_{42}}) = (2.7, 1.2)$. Meanwhile, we assume $\varepsilon = 0.25, \delta_{\theta_1} = 2.7577, \delta_{\theta_2} = 2.8231, \delta_{\theta_3} = 2.8888, \delta_{\theta_4} = 2.9547$, and $\rho = 5$. The time responses of the error variables and the adaptive parameters $\hat{k}_1, \hat{d}_1, \hat{k}_2, \hat{d}_2, \hat{k}_3, \hat{d}_3, \hat{k}_4, \hat{d}_4$ are shown in Figs. 4–6 respectively. It can be seen that the combinatorial synchronization errors e_1, e_2, e_3, e_4 converge to zero in the computable time, which means that the combinatorial inner synchronization within an anti-star sub-network with unknown parameters and external disturbances is realized in the computable time. Furthermore, all the estimated parameters $\hat{k}_i, \hat{d}_i$ $(i = 1,2,3,4)$ tend to the expected values respectively.

4.2. Combinatorial outer synchronization between different sub-networks

From the Section 4.1, we know that the parameters of all the nodes are identified, but the systems may also be disturbed by some external disturbances unavoidably. In the combinatorial outer synchronization between different sub-networks, the chain connection matrix is chosen as the exterior connection matrix, which can be written as

$$\tilde{C} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \tag{16}$$

The drive systems and the response system can be selected as

$$\begin{aligned}
\dot{x}^1 &= \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \\ \dot{w}_1 \end{pmatrix} = \underbrace{\begin{pmatrix} ax_1 - y_1z_1 + kw_1 \\ -by_1 + x_1z_1 \\ -cz_1 + x_1y_1 + dx_1 \\ -mx_1 \end{pmatrix}}_{f_1(x^1)} + \underbrace{\begin{pmatrix} 0.1\sin(x_1) \\ 0.2\sin(y_1) \\ 0.3\sin(z_1) \\ 0.4\sin(w_1) \end{pmatrix}}_{\Delta f_1(x^1)} \\
\dot{x}^2 &= \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \\ \dot{z}_2 \\ \dot{w}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} ax_2 - y_2z_2 + kw_2 \\ -by_2 + x_2z_2 \\ -cz_2 + x_2y_2 + dx_2 \\ -mx_2 \end{pmatrix}}_{f_2(x^2)} + \underbrace{\begin{pmatrix} -0.1\cos(x_2) \\ -0.2\cos(y_2) \\ -0.3\cos(z_2) \\ -0.4\cos(w_2) \end{pmatrix}}_{\Delta f_2(x^2)} + \underbrace{\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \\ w_1 - w_2 \end{pmatrix}}_{\Sigma\tilde{c}_{2,j}x^j} \\
\dot{x}^3 &= \begin{pmatrix} \dot{x}_3 \\ \dot{y}_3 \\ \dot{z}_3 \\ \dot{w}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} ax_3 - y_3z_3 + kw_3 \\ -by_3 + x_3z_3 \\ -cz_3 + x_3y_3 + dx_3 \\ -mx_3 \end{pmatrix}}_{f_3(x^3)} + \underbrace{\begin{pmatrix} 0.2\sin(x_3) \\ 0.3\cos(y_3) \\ 0.4\sin(z_3) \\ 0.5\cos(w_3) \end{pmatrix}}_{\Delta f_3(x^3)} + \underbrace{\begin{pmatrix} x_2 - x_3 \\ y_2 - y_3 \\ z_2 - z_3 \\ w_2 - w_3 \end{pmatrix}}_{\Sigma\tilde{c}_{3,j}x^j}
\end{aligned} \tag{17}$$

$$\begin{aligned}
\dot{x}^4 &= \begin{pmatrix} \dot{x}_4 \\ \dot{y}_4 \\ \dot{z}_4 \\ \dot{w}_4 \end{pmatrix} = \underbrace{\begin{pmatrix} ax_4 - y_4z_4 + kw_4 \\ -by_4 + x_4z_4 \\ -cz_4 + x_4y_4 + dx_4 \\ -mx_4 \end{pmatrix}}_{f_4(x^4)} + \underbrace{\begin{pmatrix} -0.1\cos(x_4) \\ 0.2\sin(y_4) \\ 0.3\cos(z_4) \\ -0.4\sin(w_4) \end{pmatrix}}_{\Delta f_4(x^4)} \\
&\quad + \underbrace{\begin{pmatrix} x_3 - x_4 \\ y_3 - y_4 \\ z_3 - z_4 \\ w_3 - w_4 \end{pmatrix}}_{\Sigma c_{4,j}x^j} + \underbrace{\begin{pmatrix} \tilde{u}_1(t) \\ \tilde{u}_2(t) \\ \tilde{u}_3(t) \\ \tilde{u}_4(t) \end{pmatrix}}_{\tilde{u}(t)}
\end{aligned} \tag{18}$$

During the simulation process, we take the matrices as $A_1 = diag(2,3,4,2), A_2 = diag(3,2,4,1), A_3 = diag(4,2,1,3), A_4 = diag(2,4,3,3)$. The initial conditions of the drive systems and the response system are selected as $(x_{10}, y_{10}, z_{10}, w_{10}) = (8,-7,8,12), (x_{20}, y_{20}, z_{20}, w_{20}) = (4,8,-10,1), (x_{30}, y_{30}, z_{30}, w_{30}) = (3,2,-2,5)$ and $(x_{40}, y_{40}, z_{40}, w_{40}) = (4,6,2,6)$. Thus, we have $(e_{10}, e_{20}, e_{30}, e_{40}) = (32,-25,-16,22)$. Meanwhile, we assume $\rho = 6$. The time responses of the error variables are

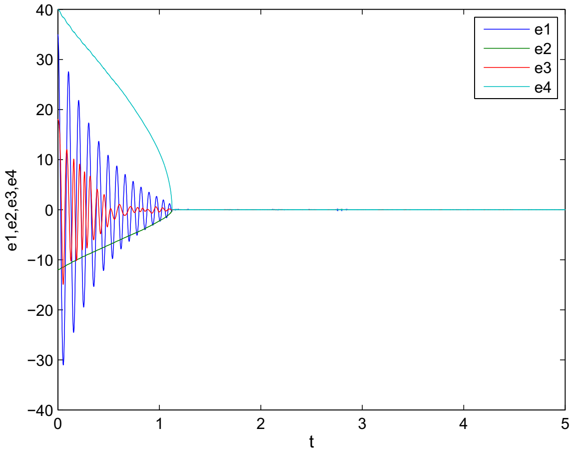

Fig. 4. Time response of the error variables e_1, e_2, e_3, e_4.

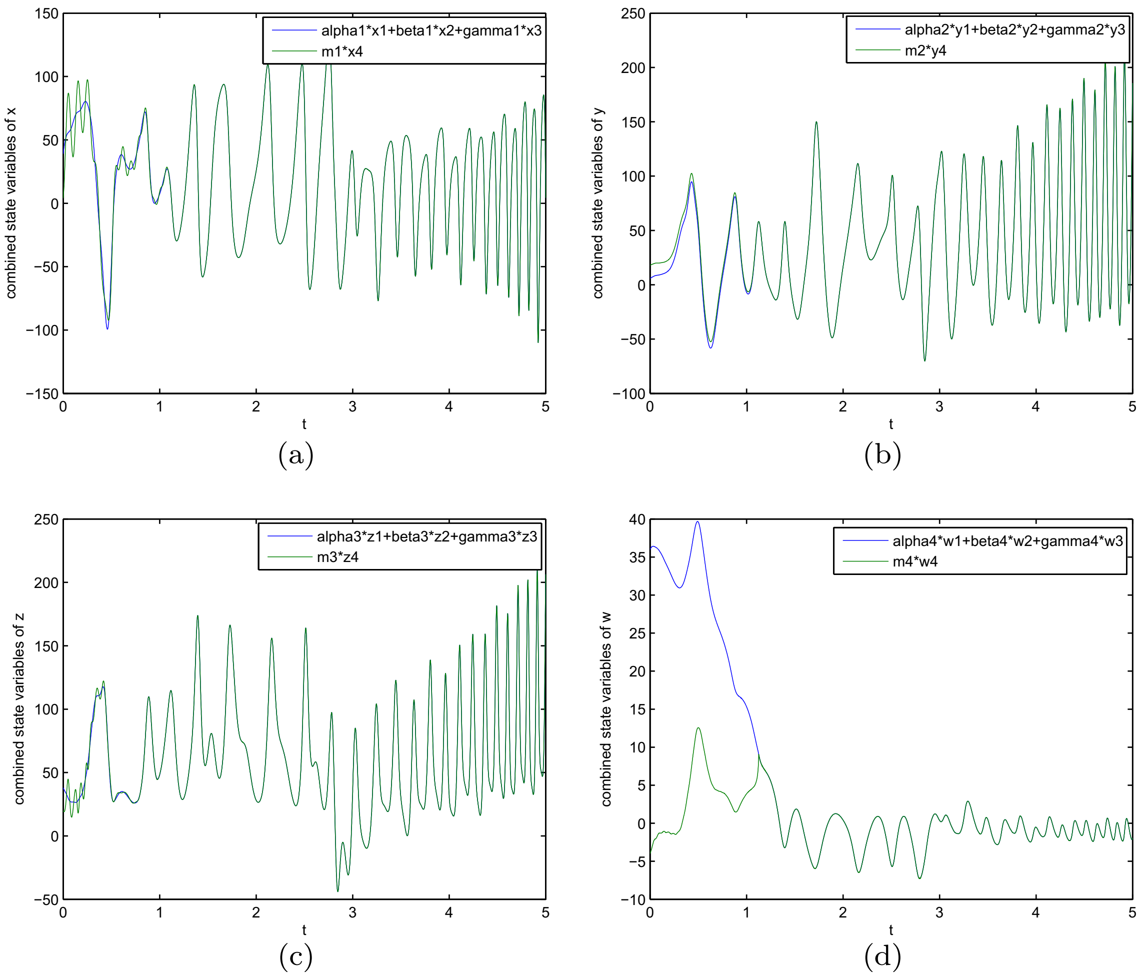

Fig. 5. Time response of the combined state variables of x, y, z, w.

shown in Figs. 7 and 8 respectively. It can be seen that the combinatorial outer synchronization errors e_1, e_2, e_3, e_4 converge to zero in the computable time, which means that the combinatorial outer synchronization between different sub-networks with external disturbances is realized in the computable time.

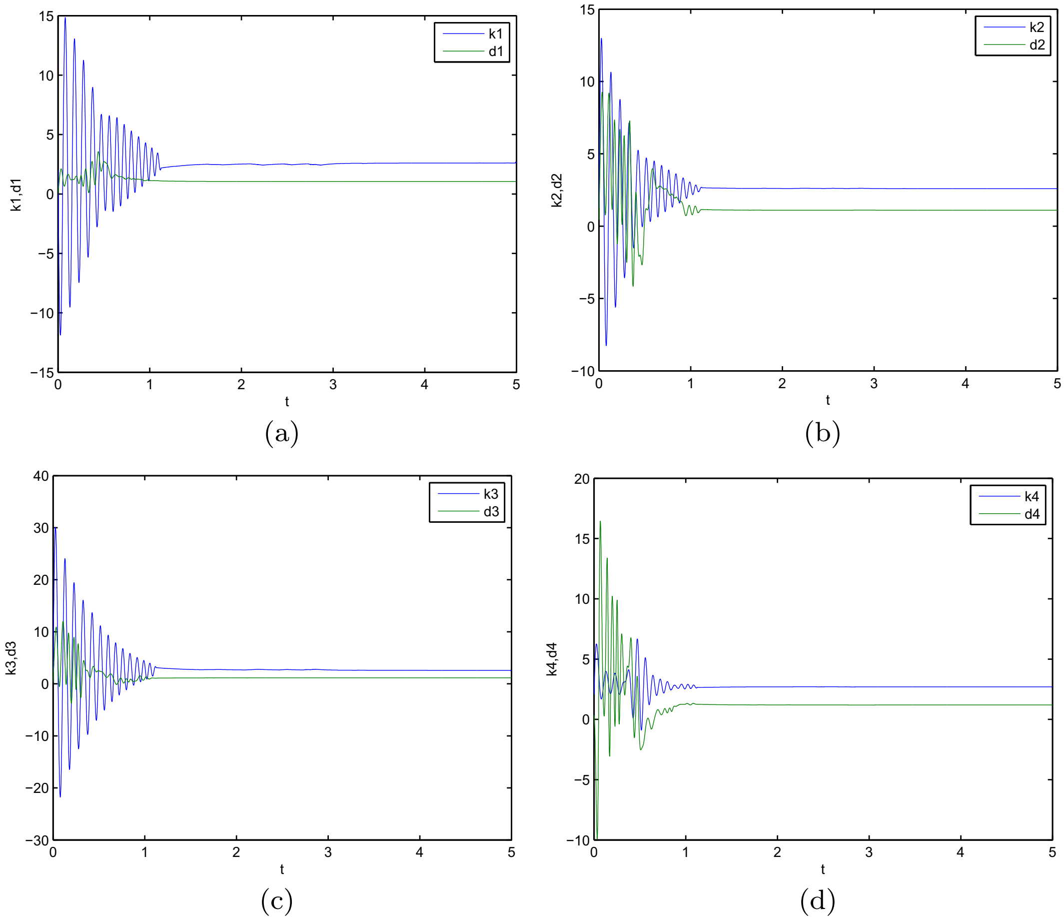

Fig. 6. Time response of the update parameters $\hat{k}_1, \hat{d}_1, \hat{k}_2, \hat{d}_2, \hat{k}_3, \hat{d}_3, \hat{k}_4, \hat{d}_4$.

5. Application of the proposed scheme in secure communication

In this section, we will test the feasibility and validity of the proposed method in the application of secure communication. As the chaotic system offers some advantages in communication systems such as broadband noise-like waveform, prediction difficulty, etc., the synchronization of chaotic systems as an effective encryption mechanism has been widely used in the secure communication [28,37]. The so-called chaotic secure communication means that one or more chaotic systems are selected as transmitter (regarded as the drive systems) and the information signal is mixed at the transmitter end to generate a chaotic transmission signal which is transmitted to the receiver end (regarded as the response system), the receiver is also a chaotic dynamic system, whose structure is based on that of the transmitter, then the information signal will be extracted by the receiver if the drive and response systems are synchronous. The level of security is mostly dependent on the complexity level of the drive's dynamics and the formation of the driving signal as well as the modulation scheme used. However, most of the typical method applied in the chaotic communication is just adding the transmitted signal to only one chaotic system, although the chaotic system with complex dynamic behavior can provide a certain degree of security, it still cannot stop the hackers, just as it is not very wise to put all your eggs in a basket. Can we transmit the information signal similar to the serial packet transport? Motivated by this, by splitting the original signal into several parts, each part with different weighting is loaded in different chaotic systems. In order to improve the security level of the transmitted information, we can select hyper-chaotic systems with much more complex dynamic behavior as the drive systems for the transmitter. Only when all parts from different sub-networks arrive at the destination in the synchronous way and the weightings as secret keys are known, will the receiver recover the signal accurately. It is qualitative to say that the method can provide dual protection for the security of the signal transmission. Furthermore, the transmitted signal is still inevitably influenced by all kinds of random factors such as channel noise, time-delay and so on, effectively reflecting these stochastic factors can help us recognize the real world more reasonably. In the following section, we will apply the combinatorial outer synchronization approach between different sub-networks with the influence of white Gaussian noise and time-delay to the secure commu-

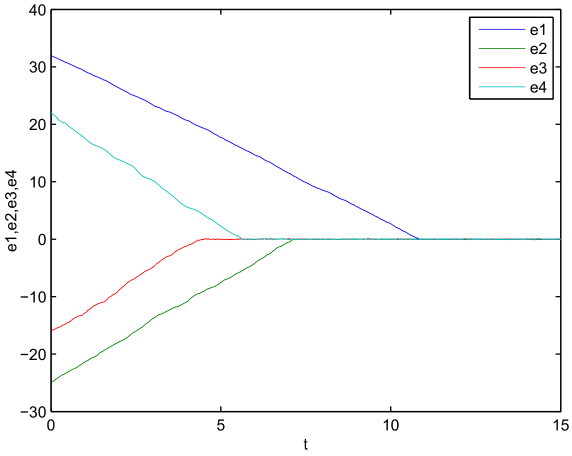

Fig. 7. Time response of the error variables e_1, e_2, e_3, e_4.

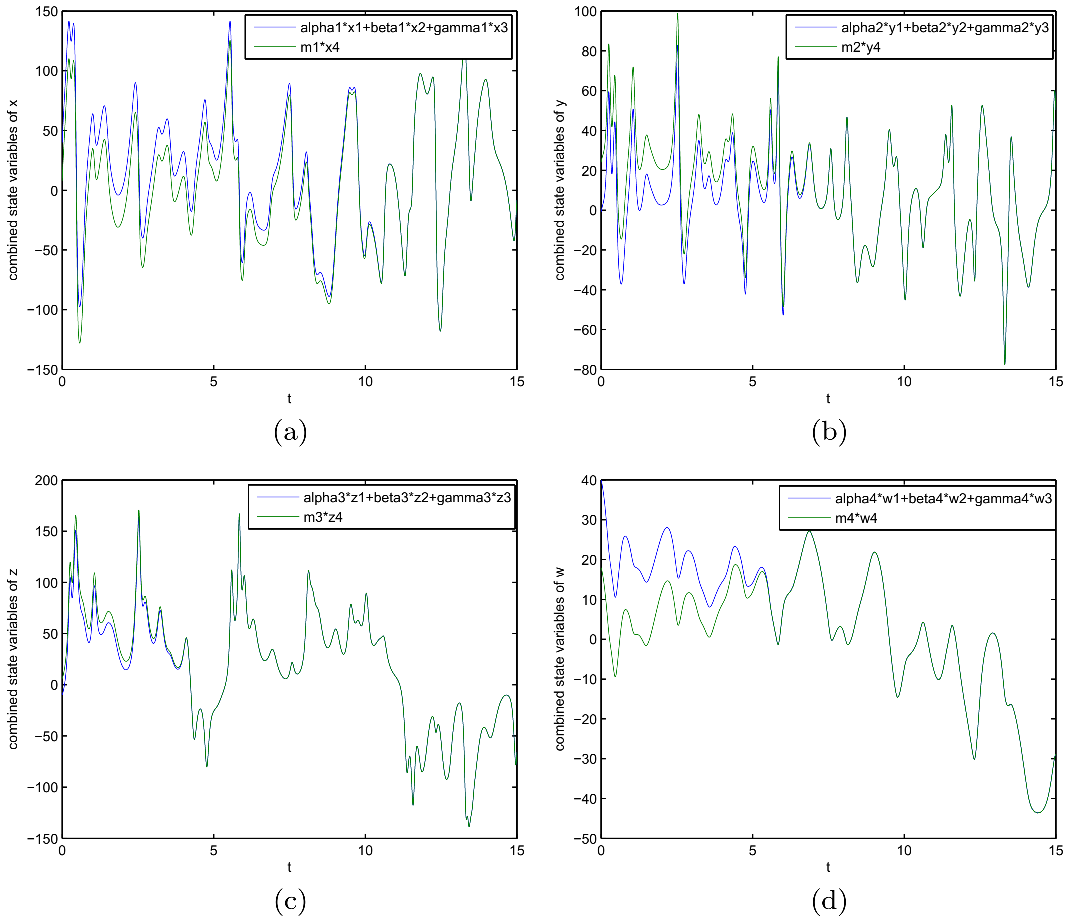

Fig. 8. Time response of the combined state variables of x, y, z, w.

nication. Assuming $m(t) = m_1(t) + m_2(t) + m_3(t)$ are the message signals which should be sent to the receiver. In order to make sure that the transmitted signals have much stronger anti-attack and anti-translated ability, we add $m_1(t), m_2(t)$ and $m_3(t)$ to the drive systems, so the state variable equations of the transmitter can be written as

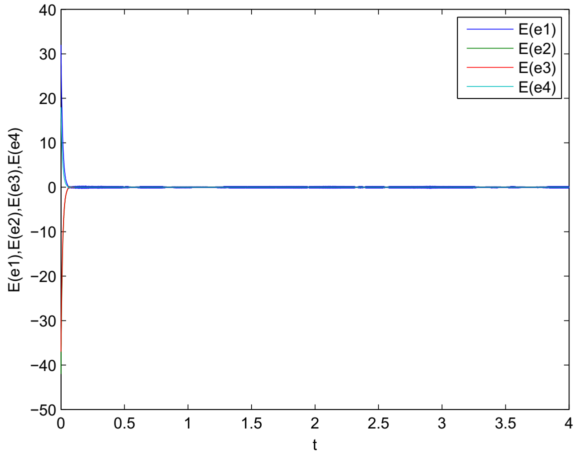

Fig. 9. Time response of the error variables $E(e_1), E(e_2), E(e_3), E(e_4)$.

$$dx^k = \left(f_k(x^k) + \Delta f_k(x^k) + \sum_{j=1}^{4} \tilde{c}_{k,j} x^j(t-\tau) + M_k\right)dt, \quad k = 1,2,3 \tag{19}$$

where M_k is a column matrix with only one nonzero element m_k and suitable dimension.

The receiver with the impact of transmission delay and stochastic noise can be constructed as

$$dx^4 = \left(f_4(x^4) + \Delta f_4(x^4) + \sum_{j=1}^{4} \tilde{c}_{4,j} x^j(t-\tau) + u(t) + s(t)\right)dt + \delta(t, e(t), e(t-\tau))d\omega \tag{20}$$

where $\delta : R^+ \times R^4 \times R^4 \to R^{4\times 4}$ is noisy intensity function, and $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)^T$ is a four-dimensional Brownian motion.

Let $e = Ax^1 + Bx^2 + Cx^3 - Dx^4, e_5 = a_4 m_1 + b_4 m_2 + c_4 m_3 - d_4 s$, where $A = diag(a_1, a_2, a_3, a_4), B = diag(b_1, b_2, b_3, b_4)$, $C = diag(c_1, c_2, c_3, c_4), D = diag(d_1, d_2, d_3, d_4)$. Before giving the Theorem 3, some assumptions must be presented.

A4. There is a constant μ such that $|a_1\dot{m}_1 + b_1\dot{m}_2 + c_1\dot{m}_3| \leqslant \mu$.

A5. The time-varying coupling delay $\tau(t)$ is a differential function with $0 \leqslant \dot{\tau}(t) \leqslant \sigma < 1$. Clearly, this hypothesis is ensured if the delay $\tau(t)$ is a constant.

A6. There are two positive constants p and q such that the noise intensity function $\delta(t, e(t), e(t-\tau))$ satisfies the condition that $trace((D\delta)^T(D\delta)) \leqslant \frac{p}{2}e^Te + \frac{q}{2}e^T(t-\tau)e(t-\tau)$. Moreover, $\delta(t, 0, 0) \equiv 0$.

For studying the convergence of random process, instead of the standard Euclidian norm, the mean square norm, L^2 norm is used, which is defined as $\|e\| = [E(e^Te)]^{\frac{1}{2}}$, where $E(\cdot)$ is the expected value function.

Theorem 3. *Let (**A1**)–(**A6**) hold and if the controller $u(t)$ is given by*

$$u(t) = D^{-1}\Bigg(Af_1(x^1) + Bf_2(x^2) + Cf_3(x^3) - Df_4(x^4) + \rho sign(e) + \left(\frac{q}{2(1-\sigma)}\int_{t-\tau}^{t} e^T(s)e(s)ds\right)^{\frac{1}{2}} \frac{e}{\|e\|^2} + A\sum_{j=1}^{4}\tilde{c}_{1,j}x^j(t-\tau) + B\sum_{j=1}^{4}\tilde{c}_{2,j}x^j(t-\tau)$$
$$+ C\sum_{j=1}^{4}\tilde{c}_{3,j}x^j(t-\tau) - D\sum_{j=1}^{4}\tilde{c}_{4,j}x^j(t-\tau) + \Gamma e + g + \frac{e}{\|e\|}\Bigg) \tag{21}$$

where $\Gamma = diag\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}(\gamma_i \in R, i = 1,2,3,4), g = (0,0,0,e_5)^T$, and the message update law of $s(t)$ is designed as

$$\dot{s}(t) = \frac{1}{d_4}\left(\mu sign(e_5) + \frac{e_5}{\|e_5\|}\right) \tag{22}$$

Then we can obtain that $E(e_i(t)) \to 0$ $(i = 1,2,3,4,5)$ in the computable time, which means that $s(t)$ can recover the message signal $m(t)$ in the computable time.

Detailed mathematical proof of Theorem 3 can be found in Appendix C of this paper.

As the message signal has been split into three parts $m_1(t), m_2(t), m_3(t)$, we add $m_1(t)$ to the right-hand side of the last equation for the first transmitter (the first drive system in Eq. (17)), $m_2(t)$ to the right-hand side of the last equation for

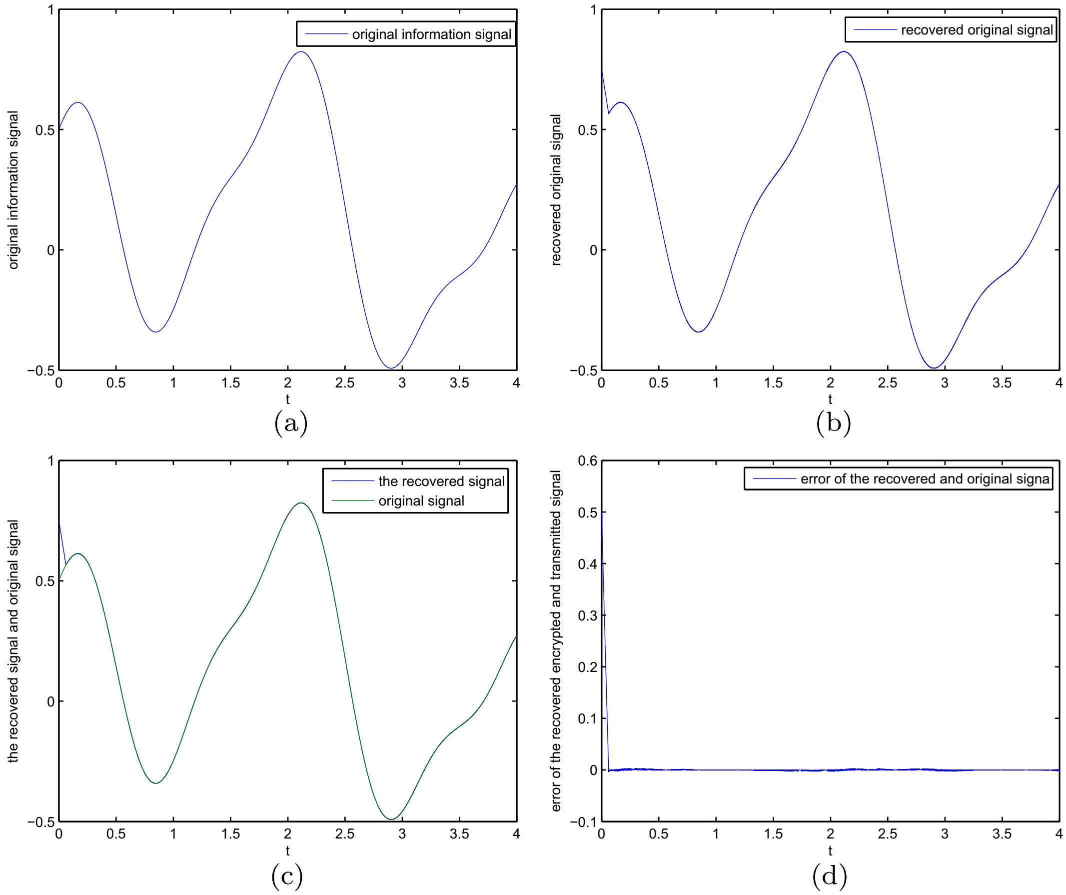

Fig. 10. Simulation results on the recovering the information signal.

the second transmitter (the second drive system in Eq. (17)) and the $m_3(t)$ to the right-hand side of the last equation for the third transmitter (the last drive system in Eq. (17)), so the drive systems of the transmitter can be modified as:

$$
\begin{aligned}
dx^1 &= \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \\ \dot{w}_1 \end{pmatrix} = \underbrace{\begin{pmatrix} ax_1 - y_1 z_1 + kw_1 \\ -by_1 + x_1 z_1 \\ -cz_1 + x_1 y_1 + dx_1 \\ -mx_1 \end{pmatrix}}_{f_1(x^1)} dt + \underbrace{\begin{pmatrix} 0.1\sin(x_1) \\ 0.2\sin(y_1) \\ 0.3\sin(z_1) \\ 0.4\sin(w_1) \end{pmatrix}}_{\Delta f_1(x^1)} dt + \begin{pmatrix} 0 \\ 0 \\ 0 \\ m_1(t) \end{pmatrix} dt \\
dx^2 &= \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \\ \dot{z}_2 \\ \dot{w}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} ax_2 - y_2 z_2 + kw_2 \\ -by_2 + x_2 z_2 \\ -cz_2 + x_2 y_2 + dx_2 \\ -mx_2 \end{pmatrix}}_{f_2(x^2)} dt + \underbrace{\begin{pmatrix} -0.1\cos(x_2) \\ -0.2\cos(y_2) \\ -0.3\cos(z_2) \\ -0.4\cos(w_2) \end{pmatrix}}_{\Delta f_2(x^2)} dt + \underbrace{\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \\ w_1 - w_2 \end{pmatrix}}_{\Sigma \tilde{c}_{2,j} x^j} dt + \begin{pmatrix} 0 \\ 0 \\ 0 \\ m_2(t) \end{pmatrix} dt \\
dx^3 &= \begin{pmatrix} \dot{x}_3 \\ \dot{y}_3 \\ \dot{z}_3 \\ \dot{w}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} ax_3 - y_3 z_3 + kw_3 \\ -by_3 + x_3 z_3 \\ -cz_3 + x_3 y_3 + dx_3 \\ -mx_3 \end{pmatrix}}_{f_3(x^3)} dt + \underbrace{\begin{pmatrix} 0.2\sin(x_3) \\ 0.3\cos(y_3) \\ 0.4\sin(z_3) \\ 0.5\cos(w_3) \end{pmatrix}}_{\Delta f_3(x^3)} dt + \underbrace{\begin{pmatrix} x_2 - x_3 \\ y_2 - y_3 \\ z_2 - z_3 \\ w_2 - w_3 \end{pmatrix}}_{\Sigma \tilde{c}_{3,j} x^j} dt + \begin{pmatrix} 0 \\ 0 \\ 0 \\ m_3(t) \end{pmatrix} dt
\end{aligned}
\tag{23}
$$

and the receiver can be constructed as

$$dx^4 = \begin{pmatrix} \dot{x}_4 \\ \dot{y}_4 \\ \dot{z}_4 \\ \dot{w}_4 \end{pmatrix} = \underbrace{\begin{pmatrix} ax_4 - y_4 z_4 + kw_4 \\ -by_4 + x_4 z_4 \\ -cz_4 + x_4 y_4 + dx_4 \\ -mx_4 \end{pmatrix}}_{f_4(x^4)} dt + \underbrace{\begin{pmatrix} -0.1\cos(x_4) \\ 0.2\sin(y_4) \\ 0.3\cos(z_4) \\ -0.4\sin(w_4) \end{pmatrix}}_{\Delta f_4(x^4)} dt + \underbrace{\begin{pmatrix} x_3 - x_4 \\ y_3 - y_4 \\ z_3 - z_4 \\ w_3 - w_4 \end{pmatrix}}_{\Sigma \tilde{c}_{4j} x^j} dt + \begin{pmatrix} 0 \\ 0 \\ 0 \\ s(t) \end{pmatrix} dt$$
$$+ \underbrace{\begin{pmatrix} \alpha_1(e_1 - e_{1\tau}) & 0 & 0 & 0 \\ 0 & \alpha_2(e_2 - e_{2\tau}) & 0 & 0 \\ 0 & 0 & \alpha_3(e_3 - e_{3\tau}) & 0 \\ 0 & 0 & 0 & \alpha_4(e_4 - e_{4\tau}) \end{pmatrix}}_{\delta(t,e(t))} \underbrace{\begin{pmatrix} d\omega_1 \\ d\omega_2 \\ d\omega_3 \\ d\omega_4 \end{pmatrix}}_{d\omega} + \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}}_{u(t)} dt \tag{24}$$

here we take $A = diag(2,3,4,2), B = diag(3,2,4,2), C = diag(4,2,1,2), D = diag(2,4,3,3), \alpha_1 = 1, \alpha_2 = -1, \alpha_3 = -2, \alpha_4 = 2,$ $\rho = 4, \gamma_1 = 75, \gamma_2 = 80, \gamma_3 = 78, \gamma_4 = 82, \mu = 5, \tau = 0.05$ and $m_1 = 0.3\sin(t), m_2 = 0.2\cos(\pi t), m_3 = 0.15\sin(2\pi t)$. The initial values of the drive systems and the response system are selected as $(x_{10}, y_{10}, z_{10}, w_{10}) = (8, -10, 6, 8), (x_{20}, y_{20}, z_{20}, w_{20}) = (4, 8, -12, 4), (x_{30}, y_{30}, z_{30}, w_{30}) = (3, 2, -4, 6)$ and $(x_{40}, y_{40}, z_{40}, w_{40}) = (4, 8, 3, 6)$. The time response of the error variables $E(e_i)$ $(i = 1, 2, 3, 4)$ is shown in Fig. 9, and the numerical simulation results on the recovering the original information signal are shown in Fig. 10. From the 9 and 10, we can easily get that $E(e_i)$ $(i = 1, 2, 3, 4, 5)$ tend to 0 in the computable time, which means that the proposed method has good immunity to the time-delay and stochastic disturbance.

6. Conclusions

In this paper, a time-controllable combinatorial inner synchronization and outer synchronization of anti-star networks, each of which consists of four-wing hyper-chaotic systems as node dynamics, is investigated. The method can realize not only the combinatorial inner synchronization within an star-like network with unknown parameters and external disturbances but also the combinatorial outer synchronization between different sub-networks with external disturbances by a switch control scheme. The switch control scheme can be set as a time trigger or an event trigger. Every sub-network has only one response system (central node in every anti-star network) as a control center to contact with not only the nodes within a sub-network but also other control centers in different sub-networks. The central node is assigned to coordinate with the other peripheral nodes, which belong to the same sub-network with the central node, to realize the combinatorial inner synchronization at first. As the fact that every sub-network can perform well alone at the same time for the combinatorial inner synchronization, and the time of the synchronization is computable, so we can set a suitable time threshold. Once it achieves at the scheduled time threshold, the control center will switch to contact with other control centers in different sub-networks for the combinatorial outer synchronization, which means that the proposed switch control scheme is fit for not only the synchronization within a network but also between different sub-networks. Furthermore, a secure communication scheme based on the adaptive combinatorial outer synchronization between different sub-networks under the influence of channel noise and time-delay is proposed. By splitting the transmitted signal into several parts, each part with different weighting is loaded in different central nodes. Only when all parts from different sub-networks arrive at the destination in the synchronous way and the weightings as secret keys are known, will the receiver recover the transmitted signal accurately in the computable time, which may have much stronger anti-attack and anti-translated ability than the traditional communication scheme with the only one-to-one transfer mode to some extent. Finally, an example on the application in the secure communication testifies the proposed method performance well even under the influence of white Gaussian noise and time-delay, which may have much more research value in the practical situation.

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Appendix A. Proof of Theorem 1

Proof. Let us consider the following Lyapunov function:

$$V_{1k}(t) = \frac{1}{2}\|e_k\|^2 + \frac{1}{2}\sum_{i=1}^{l_k}\left\|\theta_i - \hat{\theta}_i\right\|^2 \tag{25}$$

Taking the derivative on both sides of Eq. (25), and according to Eqs. (8) and (9), we can get

$$\begin{aligned}
\dot{V}_{1k}(t) &= e_k^T\dot{e}_k + \sum_{i=1}^{l_k}(\theta_i-\hat{\theta}_i)^T(-\dot{\hat{\theta}}_i) = e_k^T\Bigg(\sum_{i=1}^{l_k-1}A_if_k(x_k{}^i) - A_{l_k}f_k(x_k{}^{l_k}) + \sum_{i=1}^{l_k-1}A_iF_k(x_k{}^i)\theta_i - A_{l_k}F_k(x_k{}^{l_k})\theta_{l_k} \\
&\quad + \sum_{i=1}^{l_k-1}A_i\Delta f_i(x_k{}^i) - A_{l_k}\Delta f_{l_k}(x_k{}^{l_k}) + \sum_{j=1}^{l_k}\left(\sum_{i=1}^{l_k-1}A_ic_{ij}x_k^j - A_{l_k}c_{l_kj}x_k^j\right) - A_{l_k}u_k(t)\Bigg) \\
&\quad + \sum_{i=1}^{l_k-1}(\theta_i-\hat{\theta}_i)^T\left(-F_k^T(x_k{}^i)A_i^Te_k - \frac{\Delta\theta_i-\hat{\theta}_i}{\varepsilon}\right) + (\theta_{l_k}-\hat{\theta}_{l_k})^T\left(F_k{}^T(x_k{}^{l_k})A_{l_k}^Te_k - \frac{\Delta\theta_{l_k}-\hat{\theta}_{l_k}}{\varepsilon}\right) \\
&= e_k^T\Bigg(\sum_{i=1}^{l_k-1}A_iF_k(x_k{}^i)(\theta_i-\hat{\theta}_i) - A_{l_k}F_k(x_k{}^{l_k})\left(\theta_{l_k}-\hat{\theta}_{l_k}\right) + \sum_{i=1}^{l_k-1}A_i\Delta f_i(x_k{}^i) - A_{l_k}\Delta f_{l_k}(x_k{}^{l_k}) - \rho_k sign(e_k) - \frac{e_k}{\|e_k\|} - 2\sum_{i=1}^{l_k}\frac{\delta_{\theta_i}^2+\delta_{\theta_i}\left\|\hat{\theta}_i\right\|}{\varepsilon}\frac{e_k}{\|e_k\|^2}\Bigg) \\
&\quad + \sum_{i=1}^{l_k-1}(\theta_i-\hat{\theta}_i)^T\left(-F_k^T(x_k{}^i)A_i^Te_k - \frac{\Delta\theta_i-\hat{\theta}_i}{\varepsilon}\right) + \left(\theta_{l_k}-\hat{\theta}_{l_k}\right)^T(F_k{}^T(x_k{}^{l_k})A_{l_k}^Te_k - \frac{\Delta\theta_{l_k}-\hat{\theta}_{l_k}}{\varepsilon}) \\
&= e_k^T\left(\sum_{i=1}^{l_k-1}A_i\Delta f_i(x_k{}^i) - A_{l_k}\Delta f_{l_k}(x_k{}^{l_k}) - \rho_k sign(e_k) - \frac{e_k}{\|e_k\|} - 2\sum_{i=1}^{l_k}\frac{\delta_{\theta_i}^2+\delta_{\theta_i}\left\|\hat{\theta}_i\right\|}{\varepsilon}\frac{e_k}{\|e_k\|^2}\right) + \sum_{i=1}^{l_k}(\theta_i-\hat{\theta}_i)^T\left(-\frac{\Delta\theta_i-\hat{\theta}_i}{\varepsilon}\right) \\
&= e_k^T\left(\sum_{i=1}^{l_k-1}A_i\Delta f_i(x_k{}^i) - A_{l_k}\Delta f_{l_k}(x_k{}^{l_k}) - \rho_k sign(e_k) - \frac{e_k}{\|e_k\|} - 2\sum_{i=1}^{l_k}\frac{\delta_{\theta_i}^2+\delta_{\theta_i}\left\|\hat{\theta}_i\right\|}{\varepsilon}\frac{e_k}{\|e_k\|^2}\right) + \sum_{i=1}^{l_k}(\theta_i-\hat{\theta}_i)^T\left(\frac{\theta_i-\Delta\theta_i}{\varepsilon}\right) - \sum_{i=1}^{l_k}\frac{\left\|\theta_i-\hat{\theta}_i\right\|^2}{\varepsilon}
\end{aligned} \tag{26}$$

Using the **A1**, we can easily get the following inequalities:

$$\begin{aligned}
(\theta_i-\hat{\theta}_i)^T(\theta_i-\Delta\theta_i) &= \theta_i^T\theta_i - \theta_i^T\Delta\theta_i - \hat{\theta}_i^T\theta_i + \hat{\theta}_i^T\Delta\theta_i \leqslant \|\theta_i\|^2 + \left\|\theta_i^T\right\|\|\Delta\theta_i\| + \left\|\hat{\theta}_i^T\right\|\|\theta_i\| + \left\|\hat{\theta}_i^T\right\|\|\Delta\theta_i\| \\
&\leqslant 2\left(\delta_{\theta_i}^2 + \delta_{\theta_i}\left\|\hat{\theta}_i\right\|\right), \quad i = 1,2,\ldots,l_k
\end{aligned} \tag{27}$$

With the help of the **A2** and the inequalities (27), the Eq. (26) can be described as

$$\begin{aligned}
\dot{V}_{1k}(t) &\leqslant \|e_k^T\|\left(\left\|\sum_{i=1}^{l_k-1}A_i\Delta f_i(x_k{}^i) - A_{l_k}\Delta f_{l_k}(x_k{}^{l_k})\right\| - \rho_k\right) - \|e_k\| - 2\sum_{i=1}^{l_k}\frac{\delta_{\theta_i}^2+\delta_{\theta_i}\left\|\hat{\theta}_i\right\|}{\varepsilon} + \sum_{i=1}^{l_k}(\theta_i-\hat{\theta}_i)^T\left(\frac{\theta_i-\Delta\theta_i}{\varepsilon}\right) \\
&\quad - \sum_{i=1}^{l_k}\frac{\left\|\theta_i-\hat{\theta}_i\right\|^2}{\varepsilon} \\
&\leqslant -\|e_k\| - \sum_{i=1}^{l_k}\frac{\left\|\theta_i-\hat{\theta}_i\right\|^2}{\varepsilon}
\end{aligned} \tag{28}$$

In view of the **A3**, we have $\left\|\theta_i-\hat{\theta}_i\right\| \leqslant \frac{\left\|\theta_i-\hat{\theta}_i\right\|^2}{\varepsilon}$, which is to say that $-\frac{\left\|\theta_i-\hat{\theta}_i\right\|^2}{\varepsilon} \leqslant -\left\|\theta_i-\hat{\theta}_i\right\|$, thus we can easily get

$$\dot{V}_{1k}(t) \leqslant -\|e_k\| - \sum_{i=1}^{l_k}\left\|\theta_i-\hat{\theta}_i\right\| \tag{29}$$

According to the Lemma 2, we can get

$$\left(\|e_k\|^2 + \sum_{i=1}^{l_k}\left\|\theta_i-\hat{\theta}_i\right\|^2\right)^{\frac{1}{2}} \leqslant \|e_k\| + \sum_{i=1}^{l_k}\left\|\theta_i-\hat{\theta}_i\right\| \tag{30}$$

which is to say that inequality (29) can be written as

$$\dot{V}_{1k}(t) \leqslant -\sqrt{\|e_k\|^2 + \sum_{i=1}^{l_k}\left\|\theta_i-\hat{\theta}_i\right\|^2} = -\sqrt{2}\sqrt{\frac{1}{2}\left(\|e_k\|^2 + \sum_{i=1}^{l_k}\left\|\theta_i-\hat{\theta}_i\right\|^2\right)} = -\sqrt{2}V_{1k}^{\frac{1}{2}}(t) \tag{31}$$

Obviously, according to the Barbarlat lemma, LaSalle' invariance principle and Lemma 1, it is easy to know that the combinatorial inner synchronization error system (4) is asymptotically stable in the computable time with the largest invariant set $E = \{e_k(t) \to 0, \theta_i \to \hat{\theta}_i, i = 1,2,\ldots,l_k;\ k = 1,2,\ldots,m\}$, which means that the combinatorial inner synchronization within the *kth* sub-network can be achieved globally asymptotically in the computable time. Furthermore, the setting time T can be calculated by

$$T_k = t_0 + \sqrt{2}V_{1k}^{\frac{1}{2}}(t_0) \qquad \square$$

Appendix B. Proof of Theorem 2

Proof. Let us consider the following Lyapunov function:

$$V_2(t) = \frac{1}{2}\|\tilde{e}\|^2 \tag{32}$$

Taking the derivative on both sides of Eq. (32) and according to Eq. (10), we can obtain

$$\begin{aligned}\dot{V}_2(t) &= \tilde{e}^T\dot{\tilde{e}} = \tilde{e}^T\left(\sum_{i=1}^{m-1}\widetilde{A}_i f_i(\tilde{x}^{l_i}) - \widetilde{A}_m f_m(\tilde{x}^{l_m}) + \sum_{i=1}^{m-1}\widetilde{A}_i\Delta f_{l_i}(\tilde{x}^{l_i}) - \widetilde{A}_m\Delta f_{l_m}(\tilde{x}^{l_m}) + \sum_{j=1}^{m}\left(\sum_{i=1}^{m-1}\widetilde{A}_i\tilde{c}_{l_i,j}\tilde{x}^{l_j} - \widetilde{A}_m\tilde{c}_{l_m,j}\tilde{x}^{l_j}\right) - \widetilde{A}_m\tilde{u}(t)\right)\\ &= \tilde{e}^T\left(\sum_{i=1}^{m-1}\widetilde{A}_i\Delta f_{l_i}(\tilde{x}^{l_i}) - \widetilde{A}_m\Delta f_{l_m}(\tilde{x}^{l_m}) - \rho sign(\tilde{e}) - \frac{\tilde{e}}{\|\tilde{e}\|}\right) \leqslant \|\tilde{e}^T\|\left(\left\|\sum_{i=1}^{m-1}\widetilde{A}_i\Delta f_{l_i}(\tilde{x}^{l_i}) - \widetilde{A}_m\Delta f_{l_m}(\tilde{x}^{l_m})\right\| - \rho\right) - \|\tilde{e}\|\end{aligned} \tag{33}$$

In view of **A2**, one can obtain

$$\dot{V}_2(t) \leqslant -\|\tilde{e}\| = -\sqrt{2}\sqrt{\frac{1}{2}\|\tilde{e}\|^2} = -\sqrt{2}V_2^{\frac{1}{2}}(t) \tag{34}$$

By using the Barbarlat lemma and Lemma 1, the combinatorial outer synchronization error system (7) is asymptotically stable at the origin point $\tilde{e} = 0$ in the computable time, which is to say that the combinatorial outer synchronization between different sub-networks can also be achieved globally asymptotically in the computable time, and

$$T = t_0 + \sqrt{2}V_2^{\frac{1}{2}}(t_0) \qquad \square$$

Appendix C. Proof of Theorem 3

Proof. We first calculate the derivative of $\frac{1}{2}e_5^2$, by using the message update law of $s(t)$, we can get

$$\begin{aligned}\frac{d}{dt}\left(\frac{e_5^2}{2}\right) &= e_5\dot{e}_5 = e_5(a_4\dot{m}_1 + b_4\dot{m}_2 + c_4\dot{m}_3 - d_4\dot{s}) = e_5\left(a_4\dot{m}_1 + b_4\dot{m}_2 + c_4\dot{m}_3 - \mu sign(e_5) - \frac{e_5}{\|e_5\|}\right)\\ &\leqslant \|e_5\|(|a_4\dot{m}_1 + b_4\dot{m}_2 + c_4\dot{m}_3| - \mu - \frac{e_5}{\|e_5\|}) \leqslant -\|e_5\|\end{aligned} \tag{35}$$

Then, we choose the corresponding Lyapunov function as

$$V_3(t) = \frac{1}{2}e^Te + \frac{q}{4(1-\sigma)}\int_{t-\tau}^{t} e^T(s)e(s)ds + \frac{1}{2}e_5^2 \tag{36}$$

According to the properties about Weiner process and Itô-differential rule[38], taking the derivation of $V_3(t)$, we can get

$$\begin{aligned}dV_3 = E\Bigg[&e^T\{Af_1(x^1)+Bf_2(x^2)+Cf_3(x^3)-Df_4(x^4)+A\Delta f_1(x^1)+B\Delta f_2(x^2)+C\Delta f_3(x^3)-D\Delta f_4(x^4)+A\sum_{j=1}^{4}\tilde{c}_{1,j}x^j(t-\tau)\\ &+B\sum_{j=1}^{4}\tilde{c}_{2,j}x^j(t-\tau)+C\sum_{j=1}^{4}\tilde{c}_{3,j}x^j(t-\tau)-D\sum_{j=1}^{4}\tilde{c}_{4,j}x^j(t-\tau)-Du(t)+AM_1+BM_2+CM_3-Ds(t)\}dt+\frac{1}{2}trace((D\delta)^T(D\delta))dt\\ &+\frac{q}{4(1-\sigma)}(e^Te-(1-\dot{\tau})e^T(t-\tau)e(t-\tau))dt+e_5\dot{e}_5dt\Bigg]\end{aligned} \tag{37}$$

Substituting $u(t)$ from Eq. (21) into Eq. (37), this yields

$$\begin{aligned}dV_3 = E\Bigg[&e^T\left\{A\Delta f_1(x^1) + B\Delta f_2(x^2) + C\Delta f_3(x^3) - D\Delta f_4(x^4) - \rho sign(e) - \frac{e}{\|e\|} - \Gamma e - \left(\frac{q}{2(1-\sigma)}\int_{t-\tau}^{t} e^T(s)e(s)ds\right)^{\frac{1}{2}}\frac{e}{\|e\|^2}\right\}dt\\ &+\frac{1}{2}trace((D\delta)^T(D\delta))dt + \frac{q}{4(1-\sigma)}(e^Te - (1-\dot{\tau})e^T(t-\tau)e(t-\tau))dt + e_5\dot{e}_5dt\Bigg]\end{aligned} \tag{38}$$

In view of **A2, A4, A5** and the above inequality (35), one obtains

$$\begin{aligned}
\dot{V}_3(t) &\leqslant e^T\left(-\frac{e}{\|e\|} - \Gamma e - \left(\frac{q}{2(1-\sigma)}\int_{t-\tau}^{t} e^T(s)e(s)ds\right)^{\frac{1}{2}}\frac{e}{\|e\|^2}\right) + \frac{1}{4}\left(pe^Te + qe^T(t-\tau)e(t-\tau)\right) \\
&\quad + \frac{q}{4(1-\sigma)}\left(e^Te - (1-\dot{\tau})e^T(t-\tau)e(t-\tau)\right) - \|e_5\| \\
&= e^T\left(\left(\frac{p}{4} + \frac{q}{4(1-\sigma)}\right)I_4 - \Gamma\right)e + \left(\frac{q}{4} - \frac{q}{4(1-\sigma)}(1-\dot{\tau})\right)e^T(t-\tau)e(t-\tau) - \left(\frac{q}{2(1-\sigma)}\int_{t-\tau}^{t} e^T(s)e(s)ds\right)^{\frac{1}{2}} \\
&\quad - \|e\| - \|e_5\| \\
&\leqslant e^T\Omega e - \left(\frac{q}{2(1-\sigma)}\int_{t-\tau}^{t} e^T(s)e(s)ds\right)^{\frac{1}{2}} - \|e\| - \|e_5\|
\end{aligned} \tag{39}$$

where $\Omega = (\frac{p}{4} + \frac{q}{4(1-\sigma)})I_4 - \Gamma$ is a negative diagonal matrix with suitable diagonal elements γ_i $(i = 1,2,3,4)$. According to the Lemma 2, we can get

$$\dot{V}_3(t) \leqslant -\sqrt{2}\sqrt{\frac{q}{4(1-\sigma)}\int_{t-\tau}^{t} e^T(s)e(s)ds + \frac{1}{2}\|e\|^2 + \frac{1}{2}\|e_5\|^2} = -\sqrt{2}V_3^{\frac{1}{2}}(t) \tag{40}$$

Based on the Barbarlat lemma, Lemma 1 and LaSalle' invariance principle of stochastic differential equation, we can get $E(e_i(t)) \to 0$ $(i = 1,2,3,4,5)$ in the computable time, which means that $s(t)$ can recover the message signal $m(t)$ in the computable time and the setting time T can be defined by

$$T = t_0 + \sqrt{2}V_3^{\frac{1}{2}}(t_0) \qquad \square$$

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