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Cluster synchronization on multiple sub-networks of complex networks with nonidentical nodes via pinning control

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Abstract In this paper, cluster synchronization on multiple sub-networks of complex networks with nonidentical nodes, stochastic disturbances and time-varying delays is investigated. Based on the leader–follower model, an improved network structure model for realizing the cluster synchronization on multiple sub-networks of complex networks is presented. In this improved network model, the complex networks are divided into multiple pairs of matching sub-networks, each of which consists of a leaders' sub-network and a followers' sub-network, such that the dynamics of the nodes belonging to the same pair of matching sub-networks are identical, while the ones belonging to different pairs of unmatched sub-networks are nonidentical. Furthermore, the nodes in a sub-network may be inevitably influenced by some stochastic disturbances and time-varying delays. In this new setting, the aim is to design some suitable pinning controllers on the chosen nodes of each followers' sub-network, such that the proposed method for realizing the cluster synchronization has good immunity to the influence of these stochastic factors. By using the Lyapunov stability theory and stochastic differential equation theory, some cluster synchronization criteria, and a pinning scheme that the nodes with very large or low degrees are good candidates for applying pinning controllers, are established such that all the nodes in each sub-network are exponentially synchronized to the average state of their matching leaders. Then, the attack and robustness of the pinning scheme are discussed. Finally, some simulation examples are presented to verify the theoretical analysis.

Keywords Cluster synchronization · Multiple sub-networks of complex networks · Nonidentical node dynamics · Pinning control · Exponential stability · Stochastic disturbances and time-varying delays

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1 Introduction

Complex networks consisting of a large number of nodes and links, in which the nodes represent the individuals in the networks and the edges represent the connections among them, exist everywhere in our real world, such as biological neural networks, ecosystems, social network, the WWW, and electrical power grids. Since the small-world and scale-free complex network models were constructively proposed by Watts and Strogatz [1], Barabási and Albert [2] in 1998 and 1999, respectively, the study of various complex networks has received increasing attention from researchers in various disciplines [3]. Synchronization, as one of the most important and interesting collective behavior of complex dynamical networks, has been studied extensively due to its ubiquity in many system models, such as the large-scale complex dynamical networks [4,5], small-world neuronal networks [6,7], scale-free neuronal networks [8,9], and potential applications in many other areas, including information science, secure communication, and biological systems. Up to now, many different kinds of synchronization patterns including complete synchronization [10], global synchronization [11], stochastic synchronization [12], combinatorial synchronization [13,14], and cluster synchronization [15] have been proposed. In the case where the whole network cannot synchronize by its intrinsic structure, some control schemes may be designed to drive the network to synchronization. While, a typical real-world complex network usually consists of a large number of nodes and links, and thus, it is practically impossible to apply control actions to all nodes. To save control cost, recently, a new control method—pinning control, which means that only a small fraction of network nodes need to be applying controllers to force the whole network to synchronize, has been diffusely investigated in the synchronization of complex networks [16–19]. However, most of these pinning control schemes proposed in the previous works are just introducing a virtual centralized leader as a synchronized reference node in each network, from which a fraction of nodes in need of being controlled can receive the same information. These pinning control schemes are very fragile to the deliberate attacks; for example, if the virtual leader or even one node in the network is attacked, the whole network can be in a mess and cannot reach synchronization anymore. To overcome these shortcomings of the original framework with pinning control, it is extremely necessary and important to establish an improved pinning control scheme for the synchronization of complex networks, especially for the synchronization on multiple sub-networks of complex networks.

On the other hand, nowadays, complex networks become very huge including tens of thousands of nodes. To realize the synchronization of these large-scale complex networks, certain appropriate partition rules that can split the complex networks into multiple sub-networks should be chosen, such that the dynamics of the nodes belonging to the same sub-network are identical, while the ones belonging to different sub-networks are nonidentical. This comes to cluster synchronization on multiple sub-networks of complex networks with nonidentical nodes. The so-called cluster synchronization means that all of different sub-networks, each of which consists of identical dynamical system, can achieve synchronization individually, but typically the synchronous states of these sub-networks are mutually different, which is quite different from the "inner synchronization" [5,10] within a network and the "outer synchronization" [20–22] between two coupled complex networks. The application of cluster synchronization has been found in biological science [23,24] and communication engineering [25,26]. In view of its importance in practice, recently, studies on the cluster synchronization of complex networks have become a hot research topic. The problem of driving a general network to a selected cluster synchronization pattern by pinning control was studied in [27], and some sufficient conditions were presented to guarantee the realization of the cluster synchronization, but the considered system model is too idealistic and the nodes in a sub-network are assumed to receive the same information from the only virtual node. The projective cluster synchronization of a drive–response dynamical network with coupled partially linear chaotic systems was studied in [28], in which it is shown that the projection cluster synchronization can be realized by controlling only one node in each cluster. In [29], the cluster synchronization problem for linearly coupled network with intermittent pinning controls was investigated, and some sufficient conditions guaranteeing global cluster synchronization were presented. In order to mimic more realistic networks, Zhang et al. [30] investigated the cluster synchronization of a modified small-world networks model which possesses co-competitive weighted couplings and cluster structures, and it was proved that the new model with inter-cluster co-competition balance had an important dynamical property of robust cluster synchronous pattern formation. Nonetheless, the network nodes in these works are all considered to behave identical, and they are all assumed to synchronize to an identical node. However, it is not always practical to assume that all the network nodes are identical since some real-world complex networks may consist of different types of nodes, and these nodes always have different dynamical behaviors [31,32]. In [33], the relation between cluster synchronization and the un-weighted graph topology for nonidentical nodes was investigated, and some conditions for guaranteeing cluster synchronization were provided. The issue of mean square cluster synchronization in directed networks consisting of nonidentical nodes with communication noises was presented in

[34]. Based on the Lyapunov stability theory and stochastic theory, some sufficient synchronization conditions are derived and proved theoretically but without taking the influence of time delays into consideration.

More recently, studies on networks of networks have received increasing attention [35,36], especially since the emergence of some cooperative behaviors on networks of networks [37–41]. In [42], pinning synchronization on complex networks of networks has been investigated and some synchronization criteria are established. It should be noted that, in this paper, only the mutual cooperation of the nodes belonging to the same sub-network is taken into consideration, while the nodes may still need to interact information with the others belonging to different sub-networks in a cooperative or even competitive way. Moreover, all the nodes are assumed to be identical, and the pinning control scheme cannot be directly applied to the synchronization of complex networks with nonidentical nodes. In addition, the authors in this paper do not consider the influence of stochastic factors. However, as the fact that in the real world, due to the widespread of the random uncertainties such as stochastic forces on physical systems and noisy measurements caused by environmental uncertainties, and the finite speeds of transmission in the network, the nodes in each sub-network are often inevitably influenced by these random uncertainties and time-varying delays. Therefore, in order to make the complex networks model much more realistic, it is significant to investigate the synchronization of complex networks with the consideration of both stochastic disturbances and time-varying delays [43–48].

Based on the concept of cluster synchronization of complex networks, the results for cooperative behaviors on networks of networks and with the consideration of stochastic factors, the main contribution of this paper is that an improved framework for cluster synchronization on multiple sub-networks of complex networks via pinning control is proposed, which can be summarized as follows: (1) based on the leader–follower model, an improved network structure model that consists of multiple pairs of matching sub-networks is introduced, such that the dynamics of the nodes in each pair of matching sub-networks are identical, while the ones belonging to different pairs of unmatched sub-networks are nonidentical; (2) there are many leaders in each leaders' sub-network, from which the nodes in need of being controlled in the matching followers' sub-network can receive the information, and the average state of these leaders is regarded as the reference state; (3) the nodes belonging to the same or different sub-networks may be inevitably disturbed by some stochastic factors, and in this paper, both the random disturbances and time-varying delays are all considered; (4) in the process of information transmission, the leaders or followers can all communicate with each other in a sub-network, but only the pinned nodes can communicate with the other ones belonging to different followers' sub-networks. In addition, the pinned nodes in a followers' sub-network can receive the information from their leaders in the matching leaders' sub-network but not vice versa; and (5) the new framework for pinning cluster synchronization on multiple sub-networks of complex networks provides a certain robustness to the deliberate attacks. For example, even if some leaders or followers in a sub-network are attacked, the cluster synchronization in the other sub-networks can still be realized well, and most of the nodes in the attacked sub-network or even the whole attacked sub-network can still achieve cluster synchronization.

The rest of the paper is organized as follows: In Sect. 2, the detailed description of an improved network structure model with the consideration of stochastic factors is presented and some preliminaries are briefly outlined. Section 3 proposes some cluster synchronization criteria and a pinning control scheme for the multiple sub-networks of complex networks with nonidentical nodes. A simple analysis of the robustness for the designed pinning scheme is discussed in Sect. 4. Then, in Sect. 5, some numerical simulation examples are provided to validate all of the theoretical results. Conclusions are finally drawn in Sect. 6.

Notation Throughout this paper, some necessary notations are first introduced. $\otimes$ is the Kronecker product. $\Re^n$ represents the n-dimensional Euclidean space. $\Re^{n\times m}$ is the set of real $n \times m$ matrices. $I_N \in \Re^{N\times N}$ is an $N \times N$ identity matrix. 1_{M_k} and 0_{M_k} represent the M_k-dimensional column vectors with all the elements being 1 or 0, respectively. $\mathbf{0}$ represents a zero matrix. $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ represent the maximum and minimum eigenvalues. The superscript "T" is the transpose. $\|\cdot\|$ stands for the Euclidean vector norm. $E\{\cdot\}$ denotes the expectation. diag$\{\cdots\}$ denotes a diagonal matrix. If there are no special instructions, all of the variables are functions on t, but in this paper, we write them in a simple way for convenience, such as the nonlinear function $f(x(t))$ is equal to $f(x)$, $s_i^{(k)}(t)$ is equal to

$s_i^{(k)}$. Their corresponding time-delay variables can be simplified as well, the time-varying delay $\tau(t)$ is equal to τ_t, $e(t-\tau_t)$ is equal to e_{τ_t}, $s_i^{(k)}(t-\tau_t)$ is equal to $s_{i\tau_t}^{(k)}$, etc.

2 The formulation of the network structure model and some preliminaries

In this literature, cluster synchronization on multiple sub-networks of complex networks, which consist of multiple leaders' sub-networks and followers' sub-networks, will be studied. Based on the leader–follower model and the concept of synchronization on networks of networks presented in [35,36,42], we establish the following network structure model. The large-scale complex networks can be divided into two types of networks: the global leaders' network and the matching followers' network, which are then divided into multiple leaders' sub-networks and the matching followers' sub-networks, respectively, such that the dynamics of the nodes are identical if they belong to a pair of matching sub-networks and nonidentical if they belong to different pairs of unmatched sub-networks. In the process of information transmission, the leaders or followers can all communicate with each other in a sub-network, but only the pinned nodes can communicate with the others belonging to different followers' sub-networks. In addition, the pinned nodes in a followers' sub-network can receive the information from their leaders in the matching leaders' sub-network but not vice versa. The specific network structure diagram can be described as Fig. 1. Suppose that there is a complex dynamical network consisting of m followers' sub-networks $C_1, C_2, \ldots, C_m$ and m matching leaders' sub-networks $D_1, D_2, \ldots, D_m$, where the nodes

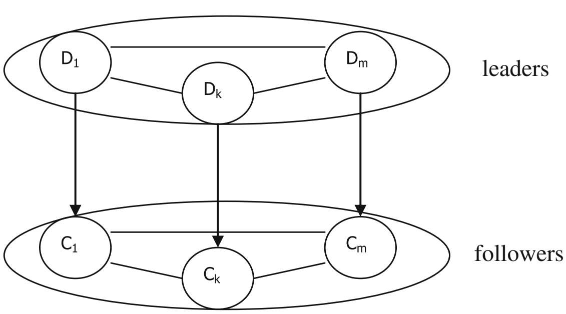

Fig. 1 The network structure diagram constructed by m leaders' sub-networks and m matching followers' sub-networks

in each followers' sub-network can receive information from their matching leaders' sub-network. Just as the network structure shown in Fig. 1, the nodes in the kth followers' sub-network can be represented as $C_k = \{r_{k-1}+1, r_{k-1}+2, \ldots, r_k\}$ and the ones in the kth leaders' sub-network can be represented as $D_k = \{w_{k-1}+1, w_{k-1}+2, \ldots, w_k\}$, where $k = 1, 2, \ldots, m$. The kth followers' sub-network has $N_k = r_k - r_{k-1}$ nodes, and the kth leaders' sub-network has $M_k = w_k - w_{k-1}$ nodes, where $r_0 = 0, r_m = N, w_0 = 0, w_m = M$; thus, we have $\sum_{k=1}^m N_k = N$ and $\sum_{k=1}^m M_k = M$. That is to say, N represents the total number of nodes in the global followers' network and M represents the total number of nodes in the global leaders' network.

Next, the cluster synchronization on multiple sub-networks of complex networks with pinning control scheme is investigated. Let $\varphi : \{1, 2, \ldots, N\}$ or $\{1, 2, \ldots, M\} \rightarrow \{1, 2, \ldots, m\}$, if node i belongs to the jth sub-network, then we have $\varphi(i) = \varphi_i = j$. Consider a complex global followers' network consisting of N nonidentical nodes, in which each node is an n-dimensional dynamic system and each node is influenced by stochastic factor, described by

$$
\begin{aligned}
\mathrm{d}x_i(t) &= [A_{\varphi_i} x_i(t) + f_{\varphi_i}(t, x_i(t), x_i(t-\tau_t))]\mathrm{d}t \\
&\quad + c_{\varphi_i} \sum_{j=1}^{N} g_{ij}^{(\varphi_i)} \Gamma x_j(t)\mathrm{d}t \\
&\quad + \delta(t, x_i(t), x_i(t-\tau_t))\mathrm{d}\omega, \quad i = 1, 2, \ldots, N \qquad (1)
\end{aligned}
$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \Re^n$ denotes the state vector of the node i in the global followers' network, $A_{\varphi_i} \in \Re^{n\times n}$ is a negative diagonal matrix, $f_{\varphi_i} : \Re \times \Re^n \times \Re^n \rightarrow \Re^n$ is a continuous vector function that describes the local dynamics of the nodes in the φ_ith followers' sub-network, $c_{\varphi_i} > 0$ stands for the coupling strength, $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \in \Re^{n\times n}$ is the inner coupling matrix and satisfies the condition $\Gamma \geq I_n$. $G^{(\varphi_i)} = (g_{ij}^{(\varphi_i)}) \in \Re^{N_{\varphi_i}\times N}$ is the coupling configuration matrix representing the topological structure of the φ_ith followers' sub-network, the precise definition of the matrix $G^{(\varphi_i)}$ can be stated as follows: (1) when $\varphi_j = \varphi_i$ and $j = i$, then $g_{ii}^{(\varphi_i)} = -\sum_{j=r_{\varphi_i-1}+1, j\neq i}^{r_{\varphi_i}} g_{ij}^{(\varphi_i)} < 0$; (2) when $\varphi_j = \varphi_i$ and $i \neq j$, and if the node i receives direct information from the node j, then $g_{ij}^{(\varphi_i)} = g_{ji}^{(\varphi_i)} > 0$; otherwise, $g_{ij}^{(\varphi_i)} = g_{ji}^{(\varphi_i)} = 0$; (3) when $\varphi_j \neq \varphi_i$, and if the node i receives direct information from the node j, then $g_{ij}^{(\varphi_i)} \neq 0$; otherwise, $g_{ij}^{(\varphi_i)} = 0$, and it satisfies the condition $\sum_{k=r_{\varphi_j-1}+1}^{r_{\varphi_j}} g_{ik}^{(\varphi_i)} = 0$. Meanwhile, $\mathrm{Deg}(i)^{(k)} = -g_{ii}^{(k)} = \sum_{j=1, j\neq i}^{N} g_{ij}^{(k)}$ represents the degree of node i in the kth sub-network. $G = [G^{(1)T}\ G^{(2)T} \ldots\ G^{(m)T}]^T \in \Re^{N\times N}$ is the coupling configuration representing the topological structure of the global followers' network, as the global followers' network considered in this paper is undirected, so the matrix G is a symmetric matrix. $\delta : \Re\times\Re^n\times\Re^n \to \Re^n$ denotes the noise intensity. τ_t represents the time-varying delay, and it satisfies $0 \leq \tau_t \leq \tau_0$, where τ_0 is a constant. $\omega(t)$ is a scalar Brownian motion, which satisfies $E\{\mathrm{d}\omega\} = 0$ and $E\{(\mathrm{d}\omega)^2\} = \mathrm{d}t$.

Remark 1 From the definition of the matrix $G^{(\varphi_i)}$, one can see that, in the same sub-network, the node i and node j are in the state of mutual cooperation; thus, if the node i receives direct information from the node j, then we have $g_{ij}^{(\varphi_i)} = g_{ji}^{(\varphi_i)} > 0$; otherwise, $g_{ij}^{(\varphi_i)} = g_{ji}^{(\varphi_i)} = 0$. However, in different sub-networks, the node i and node j may be in the state of mutual cooperation or even competition, and if the node i receives direct information from the node j, then $g_{ij}^{(\varphi_i)} > 0$ or $g_{ij}^{(\varphi_i)} < 0$, respectively; otherwise, $g_{ij}^{(\varphi_i)} = 0$, but it satisfies the condition $\sum_{k=r_{\varphi_j-1}+1}^{r_{\varphi_j}} g_{ik}^{(\varphi_i)} = 0$.

It is known to all that sometimes the network (1) may not reach synchronization by its own, and it is not too realistic to control all the nodes for realizing the synchronization of network (1). To save control cost, we can only apply some suitable controllers into just a small fraction of the nodes by using pinning control. Without loss of generality, we can rearrange the order of the nodes in the φ_ith followers' sub-network, and let the first l_{φ_i} nodes be controlled. Thus, the pinning-controlled φ_ith followers' sub-network with the influence of stochastic factors can be written as

$$\begin{aligned}\mathrm{d}x_i^{(\varphi_i)}(t) = \Bigg[& A_{\varphi_i} x_i^{(\varphi_i)}(t) + f_{\varphi_i}\left(t, x_i^{(\varphi_i)}(t), x_i^{(\varphi_i)}(t-\tau_t)\right) \\ & + c_{\varphi_i}\sum_{j=1}^{N} g_{ij}^{(\varphi_i)} \Gamma x_j(t) + u_i^{(\varphi_i)}(t)\Bigg]\mathrm{d}t \\ & + \delta\left(t, x_i^{(\varphi_i)}(t), x_i^{(\varphi_i)}(t-\tau_t)\right)\mathrm{d}\omega, \\ & i = r_{\varphi_i-1}+1, \ldots, l_{\varphi_i} \end{aligned}$$

$$\begin{aligned}\mathrm{d}x_i^{(\varphi_i)}(t) = \Bigg[& A_{\varphi_i} x_i^{(\varphi_i)}(t) + f_{\varphi_i}\left(t, x_i^{(\varphi_i)}(t), x_i^{(\varphi_i)}(t-\tau_t)\right) \\ & + c_{\varphi_i}\sum_{j=1}^{N} g_{ij}^{(\varphi_i)} \Gamma x_j(t)\Bigg]\mathrm{d}t \\ & + \delta\left(t, x_i^{(\varphi_i)}(t), x_i^{(\varphi_i)}(t-\tau_t)\right)\mathrm{d}\omega, \\ & i = l_{\varphi_i}+1, \ldots, r_{\varphi_i} \end{aligned} \tag{2}$$

where the superscript (φ_i) represents the φ_ith followers' sub-network, and $u_i^{(\varphi_i)}(t)$ is a designed controller for the φ_ith followers' sub-network.

Consider a complex global leaders' network consisting of M leaders with linearly diffusive coupling, where the dynamics of the nodes belonging to the same leaders' sub-network are identical, while the ones belonging to different leaders' sub-networks are nonidentical, and each local leaders' sub-network has $M_k (k = 1, 2, \ldots, m)$ leaders, which can be described by

$$\begin{aligned}\mathrm{d}s_i^{(k)}(t) = & \left[A_k s_i^{(k)}(t) + f_k\left(t, s_i^{(k)}(t), s_i^{(k)}(t-\tau_t)\right)\right]\mathrm{d}t \\ & + c_k \sum_{j=w_{k-1}+1}^{w_k} h_{ij}^{(k)} \Gamma s_j^{(k)}(t)\mathrm{d}t \\ & + \delta\left(t, s_i^{(k)}(t), s_i^{(k)}(t-\tau_t)\right)\mathrm{d}\omega, \\ & i = w_{k-1}+1, \ldots, w_k \end{aligned} \tag{3}$$

where $s_i^{(k)}(t) = (s_{i1}^{(k)}(t), s_{i2}^{(k)}(t), \ldots, s_{in}^{(k)}(t))^T \in \Re^n$ is the state vector of the ith leader. $H^{(k)} = (h_{ij}^{(k)}) \in \Re^{M_k\times M_k}$ is the coupling configuration matrix representing the topological structure of the kth leaders' sub-network, if the node i receives direct information from the node j, then $h_{ij}^{(k)} = h_{ji}^{(k)} \neq 0$; otherwise, $h_{ij}^{(k)} = h_{ji}^{(k)} = 0$; the diagonal elements of matrix $H^{(k)}$ are defined by $h_{ii}^{(k)} = -\sum_{j=w_{k-1}+1, j\neq i}^{w_k} h_{ij}^{(k)}$.

Prior to designing some pinning controllers for the followers' sub-networks, some assumptions must be noted as follows:

A1 There exist a constant matrix K and a positive definite matrix $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ such that f satisfies the following inequality:

$$(x-y)^T(f(x)-f(y)) \leq (x-y)^T K\Gamma(x-y), \quad \forall x, y \in \Re^n$$

A2 For the vector-valued function $f(x, \bar{x})$, there exist a constant α and a positive constant β such that

$$(x-y)^T\left[f(x,\bar{x})-f(y,\bar{y})\right] \le |\alpha|(x-y)^T(x-y) + \beta(\bar{x}-\bar{y})^T(\bar{x}-\bar{y}), \quad \forall x, y, \bar{x}, \bar{y} \in \Re^n$$

A3 There exist some positive constants $p_i, q_i (i = 1, 2, \ldots, N)$, and δ is locally Lipschitz continuous, which satisfies the linear growth condition [49]; then, we have the following condition that

$$\text{trace}\left\{\left[\delta(x_i, x_{i\tau_t}) - \delta(\bar{x}, \bar{x}_{\tau_t})\right]^T \left[\delta(x_i, x_{i\tau_t}) - \delta(\bar{x}, \bar{x}_{\tau_t})\right]\right\} \le p_i\|x_i - \bar{x}\|^2 + q_i\left\|x_{i\tau_t} - \bar{x}_{\tau_t}\right\|^2$$

A4 $\tau(t)$ is a differential function with $0 \le \dot{\tau}(t) \le \varepsilon < 1$. Clearly, this assumption is justified when $\tau(t)$ is a constant.

Now, we give the definition of the cluster exponential synchronization on multiple sub-networks of complex networks and a lemma, which is used to derive the main results.

Definition 1 A complex network with N nodes is said to realize cluster exponential synchronization, if the N nodes are split into m sub-networks $C_1, C_2, \ldots, C_m$, for arbitrary nodes i and j, if and only if there are some constants $\tilde{M}_k > 0$ and $\mu > 0$, such that for any initial conditions, inequalities $E\left\{\|x_i^{(k)}(t) - x_j^{(k)}(t)\|^2\right\} \le \tilde{M}_k\exp(-\mu t)$ hold for $t \ge 0$, where $i, j \in \{1, 2, \ldots, N\}$ and $i \ne j, k \in \{1, 2, \ldots, m\}$; then, we say that these error states converge to 0 at an exponential rate.

Lemma 1 [50] *Let x and y be arbitrary n-dimensional real vectors, let $\tilde{K}$ be a positive definite matrix, and $P \in \Re^{n\times n}$. Then, the following matrix inequality holds*:

$$2x^T P y \le x^T P \tilde{K}^{-1} P^T x + y^T \tilde{K} y$$

3 Main results for the cluster synchronization on multiple sub-networks of complex networks with pinning control scheme

In this section, with taking some stochastic disturbances and time-varying delays into consideration, we will establish some cluster synchronization criteria and propose a pinning control scheme to guarantee the realization of the exponential synchronization on multiple sub-networks of complex networks.

3.1 The cluster synchronization criteria on multiple sub-networks of complex networks with nonidentical nodes

As there are many leaders in each leaders' sub-network, in the literature, the average of all the leaders' states in a leaders' sub-network can be regarded as the reference state, and we aim to analytically prove that all the node states in each pair of matching sub-networks can be synchronized to their matching reference state. Let $\bar{s}^{(k)}(t) = \frac{1}{M_k}\sum_{j=w_{k-1}+1}^{w_k} s_j^{(k)}(t)$ be the average state of all the leaders in the kth leaders' sub-network, and as the fact that $H^{(k)} = (h_{ij}^{(k)})_{M_k\times M_k}$ is symmetric, the sum of each row is zero; thus, we can easily get that $\sum_{j=w_{k-1}+1}^{w_k}\sum_{i=w_{k-1}+1}^{w_k} h_{ji}^{(k)}\Gamma s_i^{(k)}(t) = \sum_{i=w_{k-1}+1}^{w_k}\sum_{j=w_{k-1}+1}^{w_k} h_{ji}^{(k)}\Gamma s_i^{(k)}(t) = 0$, which is to say that we have

$$\mathrm{d}\bar{s}^{(k)}(t) = \frac{1}{M_k}\sum_{j=w_{k-1}+1}^{w_k}\left\{\left[A_k s_j^{(k)}(t) + f_k\left(t, s_j^{(k)}, s_{j\tau_t}^{(k)}\right)\right]\mathrm{d}t + \delta(t, s_j^{(k)}(t), s_j^{(k)}(t-\tau_t))\mathrm{d}\omega\right\} \quad (4)$$

Let $e_j^{(sk)}(t) = s_j^{(k)}(t) - \bar{s}^{(k)}(t)$ and $e_i^{(k)}(t) = x_i^{(k)}(t) - \bar{s}^{(k)}(t)$ represent the error states from the leaders in the kth leaders' sub-network and the nodes in the kth followers' sub-network to the average state of all the leaders in the kth leaders' sub-network, respectively, where $j = w_{k-1}+1, \ldots, w_k; i = r_{k-1}+1, \ldots, r_k; k = 1, 2, \ldots, m$. Subtracting (3) from (4) yields the following error dynamical network:

$$\begin{aligned}\mathrm{d}e_i^{(sk)}(t) = &\left\{A_k s_i^{(k)}(t) + f_k\left(t, s_i^{(k)}, s_{i\tau_t}^{(k)}\right)\right.\\ &+ c_k\sum_{j=w_{k-1}+1}^{w_k} h_{ij}^{(k)}\Gamma s_j^{(k)}(t)\\ &\left.-\frac{1}{M_k}\sum_{j=w_{k-1}+1}^{w_k}\left[A_k s_j^{(k)}(t) + f_k\left(t, s_j^{(k)}, s_{j\tau_t}^{(k)}\right)\right]\right\}\mathrm{d}t\\ &+\left[\delta\left(t, s_i^{(k)}, s_{i\tau_t}^{(k)}\right) - \frac{1}{M_k}\sum_{j=w_{k-1}+1}^{w_k}\delta\left(t, s_j^{(k)}, s_{j\tau_t}^{(k)}\right)\right]\mathrm{d}\omega\\ &i = w_{k-1}+1, w_{k-1}+2, \ldots, w_k\end{aligned} \quad (5)$$

Next, two theorems are established to derive the cluster synchronization criteria for all of the leaders' sub-networks (3) and the followers' sub-networks (2), respectively.

Theorem 1 *Suppose that (A1)–(A4) hold, for given positive scalars μ, ε, $\widehat{l}_k$, the leaders in the kth leaders' sub-network are cluster exponential synchronized to their average state if there exists a symmetric and positive definite matrix $R^{(k)} \in \Re^{n\times n}$, such that the following inequalities hold*:

$$\Sigma_1^{(k)} = \Delta^{(k)} + c_k\lambda_2\left(H^{(k)}\right)I_{M_k} + 2P^{(k)} + \left[\frac{\mu}{2} + \widehat{l}_k exp(\mu\tau_t)\lambda_{\max}\left(R^{(k)}\right)\right]I_{M_k} < 0$$
$$\Sigma_3^{(k)} = \Lambda^{(k)} + 2Q^{(k)} - \widehat{l}_k(1-\varepsilon)\lambda_{\min}\left(R^{(k)}\right)I_{M_k} < 0 \tag{6}$$

where $\Delta^{(k)} = diag\{\alpha^{(k)}_{w_{k-1}+1},\ldots,\alpha^{(k)}_{w_k}\}$, $P^{(k)} = \max\{p_i\,|i = w_{k-1}+1,\ldots,w_k\}\,I_{M_k}$, $\Lambda^{(k)} = diag\{\beta^{(k)}_{w_{k-1}+1},\ldots,\beta^{(k)}_{w_k}\}$, $Q^{(k)} = \max\{q_i\,|i = w_{k-1}+1,\ldots,w_k\}I_{M_k}$. In addition, $\lambda_2(H^{(k)})$ represents the second largest eigenvalue of the matrix $H^{(k)}$, and it satisfies the condition that $\lambda_2(H^{(k)}) = \max_{x^T 1_{M_k}=0, x\neq 0_{M_k}} x^T H^{(k)}x/x^T x$ [51]. Moreover, $k = 1,2,\ldots,m$.

Proof Consider the Lyapunov functional candidate for the error system (5)

$$V_k\left(t, e^{(sk)}(t)\right) = \frac{1}{2}\sum_{i=w_{k-1}+1}^{w_k} e_i^{(sk)T}(t)e_i^{(sk)}(t)\exp(\mu t) + \widehat{l}_k\sum_{i=w_{k-1}+1}^{w_k}\int_{t-\tau_t}^{t} e_i^{(sk)T}(s)R^{(k)}e_i^{(sk)}(s) \times \exp(\mu(s+\tau_t))\mathrm{d}s \tag{7}$$

According to Itô's formula and stochastic differential equation theory [52], we take the stochastic differential $\mathrm{d}V_k(t, e^{(sk)}(t))$ along the trajectories of (5), which can be described as

$$\mathrm{d}V_k(t, e^{(sk)}(t)) = LV_k\left(t, e^{(sk)}(t)\right)\mathrm{d}t + \sum_{i=w_{k-1}+1}^{w_k} e_i^{(sk)T}(t)X_i^{(k)}\exp(\mu t)\mathrm{d}\omega \tag{8}$$

where $X_i^{(k)} = \delta(t, s_i^{(k)}, s_{i\tau_t}^{(k)}) - \frac{1}{M_k}\sum_{j=w_{k-1}+1}^{w_k}\delta(t, s_j^{(k)}, s_{j\tau_t}^{(k)})$. For convenience, we may as well let $LV_k(t, e^{(sk)}(t)) = LV_k'(t, e^{(sk)}(t))\exp(\mu t)$; then, we have

$$\begin{aligned}LV'_k(t, e^{(sk)}(t)) &= \sum_{i=w_{k-1}+1}^{w_k} e_i^{(sk)T}(t)A_k e_i^{(sk)}(t)\\
&+ \sum_{i=w_{k-1}+1}^{w_k} e_i^{(sk)T}(t)\left[f_k\left(t, s_i^{(k)}, s_{i\tau_t}^{(k)}\right) - f_k\left(t, \bar{s}^{(k)}, \bar{s}_{\tau_t}^{(k)}\right)\right]\\
&+ c_k\sum_{i=w_{k-1}+1}^{w_k} e_i^{(sk)T}(t)\sum_{j=w_{k-1}+1}^{w_k} h_{ij}^{(k)}\Gamma e_j^{(sk)}(t)\\
&- \frac{1}{M_k}\sum_{i,j=w_{k-1}+1}^{w_k} e_i^{(sk)T}(t)\left[f_k\left(t, s_j^{(k)}, s_{j\tau_t}^{(k)}\right) - f_k\left(t, \bar{s}^{(k)}, \bar{s}_{\tau_t}^{(k)}\right)\right]\\
&+ \frac{1}{2}\sum_{i=w_{k-1}+1}^{w_k}\mathrm{trace}\left(X_i^{(k)T}X_i^{(k)}\right)\\
&+ \frac{\mu}{2}\sum_{i=w_{k-1}+1}^{w_k} e_i^{(sk)T}(t)e_i^{(sk)}(t)\\
&+ \widehat{l}_k\sum_{i=w_{k-1}+1}^{w_k} e_i^{(sk)T}(t)R^{(k)}e_i^{(sk)}(t)\exp(\mu\tau_t)\\
&- \widehat{l}_k\sum_{i=w_{k-1}+1}^{w_k}(1-\dot{\tau}_t)e_{i\tau_t}^{(sk)T}R^{(k)}e_{i\tau_t}^{(sk)}\end{aligned} \tag{9}$$

In view of A3, we can easily get

$$\begin{aligned}&\frac{1}{2}\sum_{i=w_{k-1}+1}^{w_k}\mathrm{trace}\left(X_i^{(k)T}X_i^{(k)}\right)\\
&\le \frac{1}{2M_k}\sum_{i=w_{k-1}+1}^{w_k}\sum_{j=w_{k-1}+1}^{w_k} p_i\left\|s_i^{(k)}(t) - s_j^{(k)}(t)\right\|^2\\
&\quad + \frac{1}{2M_k}\sum_{i=w_{k-1}+1}^{w_k}\sum_{j=w_{k-1}+1}^{w_k} q_i\left\|s_{i\tau_t}^{(k)} - s_{j\tau_t}^{(k)}\right\|^2\\
&\le \frac{1}{M_k}\sum_{i=w_{k-1}+1}^{w_k}\sum_{j=w_{k-1}+1}^{w_k} p_i\left(\left\|e_i^{(sk)}(t)\right\|^2 + \left\|e_j^{(sk)}(t)\right\|^2\right)\\
&\quad + \frac{1}{M_k}\sum_{i=w_{k-1}+1}^{w_k}\sum_{j=w_{k-1}+1}^{w_k} q_i\left(\left\|e_{i\tau_t}^{(sk)}\right\|^2 + \left\|e_{j\tau_t}^{(sk)}\right\|^2\right)\end{aligned}$$

$$\leq \sum_{i=w_{k-1}+1}^{w_k} \left(2p^{(k)}\left\|e_i^{(sk)}(t)\right\|^2 + 2q^{(k)}\left\|e_{i\tau_t}^{(sk)}\right\|^2\right) \tag{10}$$

where $p^{(k)} = \max\{p_i\,|i = w_{k-1}+1, \ldots, w_k\}$ and $q^{(k)} = \max\{q_i\,|i = w_{k-1}+1, \ldots, w_k\}$. Using the (A1)–(A4), and with the help of the inequality (10), the Eq. (9) can be described as

$$\begin{aligned}
LV'_k\left(t, e^{(sk)}(t)\right) &\leq \sum_{i=w_{k-1}+1}^{w_k} e_i^{(sk)T}(t)A_k e_i^{(sk)}(t) \\
&+ \sum_{i=w_{k-1}+1}^{w_k} \left[\alpha_i^{(k)} e_i^{(sk)T}(t)e_i^{(sk)}(t) + \beta_i^{(k)} e_{i\tau_t}^{(sk)T} e_{i\tau_t}^{(sk)}\right] \\
&+ c_k \sum_{i=w_{k-1}+1}^{w_k} e_i^{(sk)T}(t) \sum_{j=w_{k-1}+1}^{w_k} h_{ij}^{(k)} \Gamma e_j^{(sk)}(t) \\
&+ \sum_{i=w_{k-1}+1}^{w_k} \left[2p^{(k)} e_i^{(sk)T}(t)e_i^{(sk)}(t) + 2q^{(k)} e_{i\tau_t}^{(sk)T} e_{i\tau_t}^{(sk)}\right] \\
&+ \frac{\mu}{2} \sum_{i=w_{k-1}+1}^{w_k} e_i^{(sk)T}(t)e_i^{(sk)}(t) \\
&+ \widehat{l}_k \sum_{i=w_{k-1}+1}^{w_k} e_i^{(sk)T}(t)R^{(k)}e_i^{(sk)}(t)\exp(\mu\tau_t) \\
&- \widehat{l}_k \sum_{i=w_{k-1}+1}^{w_k} (1-\varepsilon)e_{i\tau_t}^{(sk)T} R^{(k)} e_{i\tau_t}^{(sk)} \\
&= e^{(sk)T}(t)\left(I_{M_k} \otimes A_k + \Delta^{(k)} \otimes I_n\right)e^{(sk)}(t) \\
&+ e_{\tau_t}^{(sk)T}\left(\Lambda^{(k)} \otimes I_n + 2Q^{(k)} \otimes I_n\right)e_{\tau_t}^{(sk)} \\
&+ c_k e^{(sk)T}(t)\left(H^{(k)} \otimes \Gamma\right)e^{(sk)}(t) \\
&+ e^{(sk)T}(t)\left(2P^{(k)} \otimes I_n + \frac{\mu}{2} I_{M_k} \otimes I_n\right)e^{(sk)}(t) \\
&+ e^{(sk)T}(t)\left[\widehat{l}_k \exp(\mu\tau_t)\left(I_{M_k} \otimes R^{(k)}\right)\right]e^{(sk)}(t) \\
&- e_{\tau_t}^{(sk)T}\left[\widehat{l}_k(1-\varepsilon)\left(I_{M_k} \otimes R^{(k)}\right)\right]e_{\tau_t}^{(sk)} \\
&\leq e^{(sk)T}(t)\left\{\left[\Delta^{(k)} + c_k\lambda_2\left(H^{(k)}\right)I_{M_k} + 2P^{(k)}\right.\right. \\
&\left.\left. + \frac{\mu}{2} I_{M_k} + \widehat{l}_k \exp(\mu\tau_t)\lambda_{\max}(R^{(k)})I_{M_k}\right] \otimes I_n\right\} e^{(sk)}(t) \\
&+ e_{\tau_t}^{(sk)T}\left\{\left[\Lambda^{(k)} + 2Q^{(k)}\right.\right. \\
&\left.\left. - \widehat{l}_k(1-\varepsilon)\lambda_{\min}(R^{(k)})I_{M_k}\right] \otimes I_n\right\} e_{\tau_t}^{(sk)}
\end{aligned} \tag{11}$$

where $e^{(sk)} = (e_{w_{k-1}+1}^{(sk)T}, \ldots, e_{w_k}^{(sk)T})^T$, $\Delta^{(k)} = \text{diag}\{\alpha_{w_{k-1}+1}^{(k)}, \ldots, \alpha_{w_k}^{(k)}\}$, $\Lambda^{(k)} = \text{diag}\{\beta_{w_{k-1}+1}^{(k)}, \ldots, \beta_{w_k}^{(k)}\}$, $P^{(k)} = p^{(k)} I_{M_k}$, $Q^{(k)} = q^{(k)} I_{M_k}$, and $R^{(k)}$ is symmetric and positive definite. Let $\zeta^{(k)}(t) = \left(e^{(sk)T}(t)\ \ e_{\tau_t}^{(sk)T}\right)^T$ and

$$\Omega^{(k)} = \begin{bmatrix} \Sigma_1^{(k)} & \Sigma_2^{(k)} \\ \Sigma_2^{(k)T} & \Sigma_3^{(k)} \end{bmatrix} \tag{12}$$

where $\Sigma_1^{(k)} = [\Delta^{(k)} + c_k\lambda_2(H^{(k)})I_{M_k} + 2P^{(k)} + \frac{\mu}{2} I_{M_k} + \widehat{l}_k \exp(\mu\tau_t)\lambda_{\max}(R^{(k)})I_{M_k}] \otimes I_n < 0$, $\Sigma_3^{(k)} = [\Lambda^{(k)} + 2Q^{(k)} - \widehat{l}_k(1-\varepsilon)\lambda_{\min}(R^{(k)})I_{M_k}] \otimes I_n < 0$, and $\Sigma_2^{(k)} = \Sigma_2^{(k)T} = \mathbf{0}$. Therefore, we have $\Omega^{(k)} < 0$. Taking the mathematical expectation on both sides of (8) and considering (11), we have

$$\frac{\mathrm{d}E\left\{V_k(t, e^{(sk)}(t))\right\}}{\mathrm{d}t} \leq E\left\{\zeta^{(k)T}(t)\Omega^{(k)}\zeta^{(k)}(t)\right\} \times \exp(\mu t) \tag{13}$$

Note from (12) that $\Omega^{(k)} < 0$, then, one can further deduce that

$$\frac{\mathrm{d}E\left\{V_k(t, e^{(sk)}(t))\right\}}{\mathrm{d}t} \leq -\lambda_{\min}(-\Omega^{(k)})E\left\{\left\|e^{(sk)}(t)\right\|^2\right\}\exp(\mu t) \tag{14}$$

It follows from (14) that $E\{V_k(t, e^{(sk)})\} \leq E\{V_k(0, e_0^{(sk)})\}(t \geq 0)$, which implies

$$\begin{aligned}
E\left\{\left\|e^{(sk)}(t)\right\|^2\right\} &\leq 2E\left\{V_k\left(t, e^{(sk)}(t)\right)\right\}\exp(-\mu t) \\
&\leq 2E\left\{V_k\left(0, e_0^{(sk)}\right)\right\}\exp(-\mu t)
\end{aligned} \tag{15}$$

Obviously, it can be easily found that there is a positive scalar $\tilde{M}_k$, such that $2V_k(0, e_0^{(sk)}) \leq \tilde{M}_k$, which is to say, we have $E\left\{\left\|e^{(sk)}(t)\right\|^2\right\} \leq \tilde{M}_k\exp(-\mu t)$. Thus, all the leaders in the kth leaders' sub-network are exponentially synchronized to their matching reference state, where $k = 1, 2, \ldots, m$, this completes the proof. □

The dynamics of the reference states satisfy

$$\begin{aligned}
\mathrm{d}\bar{s}^{(k)}(t) &= \left[A_k\bar{s}^{(k)}(t) + f_k\left(t, \bar{s}^{(k)}(t), \bar{s}^{(k)}(t-\tau_t)\right)\right]\mathrm{d}t \\
&+ \delta\left(t, \bar{s}^{(k)}(t), \bar{s}^{(k)}(t-\tau_t)\right)\mathrm{d}\omega, \\
&k = 1, 2, \ldots, m
\end{aligned} \tag{16}$$

Since $e_i^{(k)}(t) = x_i^{(k)}(t) - \bar{s}^{(k)}(t), k = 1, 2, \ldots, m$, and let $e_i(t) = \begin{cases} e_i^{(1)}(t), \varphi_i = 1 \\ e_{i-N_{k-1}}^{(k)}(t), \varphi_i = k, k \neq 1 \end{cases}$, one can have the following error system between the nodes in a followers' sub-network and their matching reference state:

$$\begin{aligned} \mathrm{d}e_i^{(k)}(t) = \Bigg[& A_k e_i^{(k)}(t) + f_k\left(t, x_i^{(k)}, x_{i\tau_t}^{(k)}\right) \\ & - f_k\left(t, \bar{s}^{(k)}, \bar{s}_{\tau_t}^{(k)}\right) \\ & + c_k \sum_{j=1}^{N} g_{ij}^{(k)} \Gamma e_j(t) + u_i^{(k)}(t) \Bigg] \mathrm{d}t \\ & + \left[\delta\left(t, x_i^{(k)}, x_{i\tau_t}^{(k)}\right) - \delta\left(t, \bar{s}^{(k)}, \bar{s}_{\tau_t}^{(k)}\right)\right] \mathrm{d}\omega \\ & i = r_{k-1}+1, r_{k-1}+2, \ldots, r_k \end{aligned} \tag{17}$$

Theorem 2 *Let assumptions (A1)–(A4) hold, and the controller $u_i^{(k)}(t)$ is given by*

$$u_i^{(k)}(t) = -c_k \sum_{j=w_{k-1}+1}^{w_k} d_{ij}^{(k)} \Gamma \left(x_i^{(k)}(t) - s_j^{(k)}(t)\right) \tag{18}$$

where all the control gains $d_{ij}^{(k)} > 0$ and $i = r_{k-1}+1, \ldots, l_k; k = 1, 2, \ldots, m$. Then, the controlled network (2) is exponentially synchronized if there exists a symmetric and positive definite matrix $\tilde{R} \in \Re^{n\times n}$, and the following conditions are satisfied:

$$\begin{aligned} \tilde{\Sigma}_1 = & \tilde{\Delta} + \tilde{C}G - \tilde{C}D + \frac{1}{2}\tilde{P} + \frac{\mu}{2} I_N \\ & + \tilde{l} exp(\mu\tau_t)\lambda_{\max}(\tilde{R}) I_N < 0 \\ \tilde{\Sigma}_3 = & \tilde{\Lambda} + \frac{1}{2}\tilde{Q} - \tilde{l}(1-\varepsilon)\lambda_{\min}(\tilde{R}) I_N < 0 \end{aligned} \tag{19}$$

where $\mu, \varepsilon, \tilde{l} > 0$, $\tilde{\Delta}^{(k)} = diag\{\alpha_{r_{k-1}+1}^{(k)}, \ldots, \alpha_{r_k}^{(k)}\}$, $\tilde{\Lambda}^{(k)} = diag\{\beta_{r_{k-1}+1}^{(k)}, \ldots, \beta_{r_k}^{(k)}\}$, $\tilde{P}^{(k)} = diag\{p_{r_{k-1}+1}^{(k)}, \ldots, p_{r_k}^{(k)}\}$, $\tilde{Q}^{(k)} = diag\{q_{r_{k-1}+1}^{(k)}, \ldots, q_{r_k}^{(k)}\}$, $\tilde{\Delta} = diag\{\tilde{\Delta}^{(1)}, \ldots, \tilde{\Delta}^{(m)}\}$, $\tilde{\Lambda} = diag\{\tilde{\Lambda}^{(1)}, \ldots, \tilde{\Lambda}^{(m)}\}$, $\tilde{P} = diag\{\tilde{P}^{(1)}, \ldots, \tilde{P}^{(m)}\}$, $\tilde{Q} = diag\{\tilde{Q}^{(1)}, \ldots, \tilde{Q}^{(m)}\}$, $D = diag\left\{D^{(1)}, \ldots, D^{(m)}\right\}$, and $\tilde{C} = diag\{c_1 I_{N_1}, \ldots, c_m I_{N_m}\}$, where $D^{(k)} = diag\{\underbrace{\sum_{j=w_{k-1}+1}^{w_k} d_{r_{k-1}+1,j}^{(k)}, \ldots, \sum_{j=w_{k-1}+1}^{w_k} d_{l_k,j}^{(k)}}_{l_k}, \underbrace{0, \ldots, 0}_{N_k - l_k}\}$.

Proof Since $e_j^{(sk)}(t)$ are independent of $e_i^{(k)}(t)$, where $j = w_{k-1}+1, \ldots, w_k; i = r_{k-1}+1, \ldots, r_k$ and $k = 1, 2, \ldots, m$, the synchronization of all the nodes in the leaders' sub-network (3) has been established; then, we consider the Lyapunov functional candidate for the error system (17)

$$\begin{aligned} V(t, e(t)) = & \frac{1}{2} \sum_{k=1}^{m} \sum_{i=r_{k-1}+1}^{r_k} e_i^{(k)T}(t) e_i^{(k)}(t) \exp(\mu t) \\ & + \tilde{l} \sum_{k=1}^{m} \sum_{i=r_{k-1}+1}^{r_k} \int_{t-\tau_t}^{t} e_i^{(k)T}(s) \tilde{R} e_i^{(k)}(s) \\ & \times \exp(\mu(s+\tau_t)) \mathrm{d}s \end{aligned} \tag{20}$$

Based on the stochastic differential equation theory [52], and taking the stochastic differential $\mathrm{d}V(t, e(t))$ along the trajectories of (17), we have

$$\begin{aligned} \mathrm{d}V(t, e(t)) = & LV(t, e(t)) \mathrm{d}t \\ & + \sum_{k=1}^{m} \sum_{i=r_{k-1}+1}^{r_k} e_i^{(k)T}(t) Y_i^{(k)} \exp(\mu t) \mathrm{d}\omega \end{aligned} \tag{21}$$

where $Y_i^{(k)} = \delta(t, x_i^{(k)}, x_{i\tau_t}^{(k)}) - \delta(t, \bar{s}^{(k)}, \bar{s}_{\tau_t}^{(k)})$.
For convenience, we may as well let $LV(t, e(t)) = LV'(t, e(t))\exp(\mu t)$, and where

$$\begin{aligned} LV'(t, e(t)) = & \sum_{k=1}^{m} \sum_{i=r_{k-1}+1}^{r_k} e_i^{(k)T}(t) A_k e_i^{(k)}(t) \\ & + \sum_{k=1}^{m} \sum_{i=r_{k-1}+1}^{r_k} e_i^{(k)T}(t) \left[f_k\left(t, x_i^{(k)}, x_{i\tau_t}^{(k)}\right) \right. \\ & \left. - f_k\left(t, \bar{s}^{(k)}, \bar{s}_{\tau_t}^{(k)}\right) \right] \\ & + \sum_{k=1}^{m} c_k \sum_{i=r_{k-1}+1}^{r_k} e_i^{(k)T}(t) \sum_{j=1}^{N} g_{ij}^{(k)} \Gamma e_j(t) \\ & - \sum_{k=1}^{m} c_k \sum_{i=r_{k-1}+1}^{l_k} e_i^{(k)T}(t) \sum_{j=w_{k-1}+1}^{w_k} d_{ij}^{(k)} \Gamma e_i^{(k)}(t) \\ & + \frac{1}{2} \sum_{k=1}^{m} \sum_{i=r_{k-1}+1}^{r_k} \operatorname{trace}\left(Y_i^{(k)T} Y_i^{(k)}\right) \\ & + \frac{\mu}{2} \sum_{k=1}^{m} \sum_{i=r_{k-1}+1}^{r_k} e_i^{(k)T}(t) e_i^{(k)}(t) \\ & + \tilde{l} \sum_{k=1}^{m} \sum_{i=r_{k-1}+1}^{r_k} e_i^{(k)T}(t) \tilde{R} \exp(\mu\tau_t) e_i^{(k)}(t) \\ & - \tilde{l} \sum_{k=1}^{m} \sum_{i=r_{k-1}+1}^{r_k} (1-\dot{\tau}_t) e_{i\tau_t}^{(k)T} \tilde{R} e_{i\tau_t}^{(k)} \end{aligned} \tag{22}$$

In view of (A1)–(A4), one can obtain

$$\begin{aligned} LV'(t,e(t)) \le & \sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{r_k} e_i^{(k)T}(t)A_k e_i^{(k)}(t) \\ & +\sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{r_k}\left[\alpha_i^{(k)}e_i^{(k)T}(t)e_i^{(k)}(t)+\beta_i^{(k)}e_{i\tau_t}^{(k)T}e_{i\tau_t}^{(k)}\right] \\ & +\sum_{k=1}^{m}c_k\sum_{i=r_{k-1}+1}^{r_k}e_i^{(k)T}(t)\sum_{j=1}^{N}g_{ij}^{(k)}\Gamma e_j(t) \\ & -\sum_{k=1}^{m}c_k\sum_{i=r_{k-1}+1}^{l_k}e_i^{(k)T}(t)\sum_{j=w_{k-1}+1}^{w_k}d_{ij}^{(k)}\Gamma e_i^{(k)}(t) \\ & +\frac{1}{2}\sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{r_k}\left[p_i^{(k)}e_i^{(k)T}(t)e_i^{(k)}(t)+q_i^{(k)}e_{i\tau_t}^{(k)T}e_{i\tau_t}^{(k)}\right] \\ & +\frac{\mu}{2}\sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{r_k}e_i^{(k)T}(t)e_i^{(k)}(t) \\ & +\tilde{l}\sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{r_k}e_i^{(k)T}(t)\tilde{R}\exp(\mu\tau_t)e_i^{(k)}(t) \\ & -\tilde{l}\sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{r_k}(1-\varepsilon)e_{i\tau_t}^{(k)T}\tilde{R}e_{i\tau_t}^{(k)} \end{aligned} \tag{23}$$

Let $e^{(k)}(t)=\left(e_{r_{k-1}+1}^{(k)T}(t)\ldots e_{r_k}^{(k)T}(t)\right)^T$, it follows that

$$\begin{aligned} LV'(t,e(t)) \le & \sum_{k=1}^{m}e^{(k)T}(t)\left(I_{N_k}\otimes A_k\right. \\ & \left.+\tilde{\Delta}^{(k)}\otimes I_n\right)e^{(k)}(t) \\ & +\sum_{k=1}^{m}e_{\tau_t}^{(k)T}\left(\tilde{\Lambda}_k\otimes I_n\right)e_{\tau_t}^{(k)} \\ & +\sum_{k=1}^{m}c_k e^{(k)T}(t)\left(G^{(k)}\otimes\Gamma\right)e(t) \\ & -\sum_{k=1}^{m}c_k e^{(k)T}(t)\left(D^{(k)}\otimes\Gamma\right)e^{(k)}(t) \\ & +\sum_{k=1}^{m}e_{\tau_t}^{(k)T}\left(\frac{1}{2}\tilde{Q}^{(k)}\otimes I_n\right)e_{\tau_t}^{(k)} \\ & +\frac{\mu}{2}\sum_{k=1}^{m}e^{(k)T}(t)e^{(k)}(t) \\ & +\sum_{k=1}^{m}e^{(k)T}(t)\left[\frac{1}{2}\tilde{P}^{(k)}\otimes I_n\right. \\ & \left.+\tilde{l}\exp(\mu\tau_t)I_{N_k}\otimes\tilde{R}\right]e^{(k)}(t) \\ & -\sum_{k=1}^{m}e_{\tau_t}^{(k)T}\left[\tilde{l}(1-\varepsilon)I_{N_k}\otimes\tilde{R}\right]e_{\tau_t}^{(k)} \end{aligned} \tag{24}$$

where $\tilde{\Delta}^{(k)}=\text{diag}\{\alpha_{r_{k-1}+1}^{(k)},\ldots,\alpha_{r_k}^{(k)}\}$, $\tilde{\Lambda}^{(k)}=\text{diag}\{\beta_{r_{k-1}+1}^{(k)},\ldots,\beta_{r_k}^{(k)}\}$, $\tilde{P}^{(k)}=\text{diag}\{p_{r_{k-1}+1}^{(k)},\ldots,p_{r_k}^{(k)}\}$, $\tilde{Q}^{(k)}=\text{diag}\{q_{r_{k-1}+1}^{(k)},\ldots,q_{r_k}^{(k)}\}$, and $k=1,2,\ldots,m$. $D^{(k)}=\text{diag}\{\underbrace{\sum_{j=w_{k-1}+1}^{w_k}d_{r_{k-1}+1,j}^{(k)},\ldots,\sum_{j=w_{k-1}+1}^{w_k}d_{l_k,j}^{(k)}}_{l_k},\underbrace{0,\ldots,0}_{N_k-l_k}\}$, and $\sum_{j=w_{k-1}+1}^{w_k}d_{ij}^{(k)}$ is the total weights of the ith node in the kth followers' sub-network receiving from the leaders in the kth leaders' sub-network. Without loss of generality, let $e(t)=\left(e_1^T(t)\ldots e_N^T(t)\right)^T=\left(e^{(1)T}(t)\ldots e^{(m)T}(t)\right)^T$, $e_{\tau_t}=\left(e_{\tau_t}^{(1)T}\cdots e_{\tau_t}^{(m)T}\right)^T$, and $A=\text{diag}\{I_{N_1}\otimes A_1,\ldots,I_{N_m}\otimes A_m\}$, $\tilde{\Delta}=\text{diag}\{\tilde{\Delta}^{(1)},\ldots,\tilde{\Delta}^{(m)}\}$, $\tilde{\Lambda}=\text{diag}\{\tilde{\Lambda}^{(1)},\ldots,\tilde{\Lambda}^{(m)}\}$, $D=\text{diag}\{D^{(1)},\ldots,D^{(m)}\}$, $\tilde{P}=\text{diag}\{\tilde{P}^{(1)},\ldots,\tilde{P}^{(m)}\}$, $\tilde{Q}=\text{diag}\{\tilde{Q}^{(1)},\ldots,\tilde{Q}^{(m)}\}$, $\tilde{C}=\text{diag}\left\{c_1I_{N_1},\ldots,c_mI_{N_m}\right\}$. Then, inequality (24) can be written as

$$\begin{aligned} LV'(t,e(t)) \le & e^T(t)Ae(t)+e^T(t)(\tilde{\Delta}\otimes I_n)e(t) \\ & +e_{\tau_t}^T\left(\tilde{\Lambda}\otimes I_n\right)e_{\tau_t}+e^T(t)\left(\tilde{C}G\otimes\Gamma\right)e(t) \\ & -e^T(t)\left(\tilde{C}D\otimes\Gamma\right)e(t)+\frac{1}{2}e^T(t)\left(\tilde{P}\otimes I_n\right)e(t) \\ & +\frac{1}{2}e_{\tau_t}^T\left(\tilde{Q}\otimes I_n\right)e_{\tau_t}+e^T(t)\left(\frac{\mu}{2}I_N\otimes I_n\right)e(t) \\ & +e^T(t)\left[\tilde{l}\exp(\mu\tau_t)I_N\otimes\tilde{R}\right]e(t) \\ & -e_{\tau_t}^T\left[\tilde{l}(1-\varepsilon)I_N\otimes\tilde{R}\right]e_{\tau_t} \\ \le & e^T(t)\left[\left(\tilde{\Delta}+\tilde{C}G-\tilde{C}D+\frac{1}{2}\tilde{P}+\frac{\mu}{2}I_N\right.\right. \\ & \left.\left.+\tilde{l}\exp(\mu\tau_t)\lambda_{\max}(\tilde{R})I_N\right)\otimes\Gamma\right]e(t) \\ & +e_{\tau_t}^T\left[\left(\tilde{\Lambda}+\frac{1}{2}\tilde{Q}-\tilde{l}(1-\varepsilon)\lambda_{\min}(\tilde{R})I_N\right)\otimes I_n\right]e_{\tau_t} \\ = & \zeta^T\tilde{\Omega}\zeta \end{aligned} \tag{25}$$

where $\zeta(t)=\left[e^T(t)\ e_{\tau_t}^T\right]^T$, and $\tilde{\Omega}=\begin{bmatrix}\tilde{\Sigma}_1 & \tilde{\Sigma}_2\\ \tilde{\Sigma}_2^T & \tilde{\Sigma}_3\end{bmatrix}$ is symmetric and negative definite, in which $\tilde{\Sigma}_1=[\tilde{\Delta}+\tilde{C}G-\tilde{C}D+\frac{1}{2}\tilde{P}+\frac{\mu}{2}I_N+\tilde{l}\exp(\mu\tau_t)\lambda_{\max}(\tilde{R})I_N]\otimes\Gamma<0$, $\tilde{\Sigma}_2=\tilde{\Sigma}_2^T=\mathbf{0}$ and $\tilde{\Sigma}_3=[\tilde{\Lambda}+\frac{1}{2}\tilde{Q}-\tilde{l}(1-\varepsilon)\lambda_{\min}(\tilde{R})I_N]\otimes I_n<0$. That is to say, we can easily get that $\tilde{\Omega} < 0$. The rest procedure of the proof is the same as that of in Theorem 1; hence, we omit it here. This completes the proof. □

From the conditions (6) in Theorem 1 and (19) in Theorem 2, it is easy to observe that the networks (2) and (3) are exponentially synchronized under the given linear feedback pinning controller (18), where $k = 1, 2, \dots, m$.

Remark 2 Theorems 1 and 2 are very intuitive, from the first formula in the conditions (6) and (19), we can see that the larger the coupling strength is, the easier these conditions are satisfied. In particular, if $M \equiv 1$ and $m = 1$, then the present framework is the original pinning synchronization of complex networks with identical node; if $M_k \equiv 1$ and $k = 1, 2, \dots, m$, then it becomes the traditional cluster synchronization of complex networks with nonidentical nodes via pinning control. The advantages of the proposed network model and pinning control scheme can be summarized as follows: (1) based on the leader–follower model, an improved network model is presented, which is much more realistic than most of the previous network models even the network model proposed recently in [42]. In the improved network model, the complex networks are composed of multiple pairs of matching sub-networks, and the nodes in each pair of matching sub-networks are assumed to be identical, while the ones belonging to different pairs of unmatched sub-networks are assumed to be nonidentical. This is superior to the assumption of all nodes with identical node dynamics in the whole network [42], and the network structure in Ref. [42] cannot be directly applied to the complex networks with nonidentical nodes; (2) there are many leaders in each leaders' sub-network, from which the pinned nodes in the matching followers' sub-network can receive the information, and the average state of all the leaders in each leaders' sub-network is regarded as the reference state, instead of a homogeneous state; (3) in the process of information transmission, the leaders or followers can all communicate with each other in a sub-network, but only the pinned nodes can communicate with the others belonging to different followers' sub-networks; in addition, the pinned nodes in a followers' sub-network can receive the information from their leaders in the matching leaders' sub-network but not vice versa; and (4) the improved framework for pinning cluster synchronization on multiple sub-networks of complex networks is testified to have good robustness to the influence of stochastic factors and deliberate attacks. For example, even if some leaders or followers in a sub-network are attacked, the other sub-networks can reach cluster synchronization well; furthermore, most of the nodes in the attacked sub-network or even the whole attacked sub-network can still achieve cluster synchronization.

Remark 3 The theoretical coupling strengths given in (6) and (19) are conservative, which are usually much larger than the needed values. In fact, it is desirable to make the coupling strength as small as possible, so suitable adaptive technique can be adopted to achieve this goal, which will be one part of our next work.

Remark 4 The proposed network structure model is much closer to the reality, which can be widely exist in social networks, such as enterprise management networks and the teaching networks between the teachers and the students.

3.2 The pinning control scheme for the cluster synchronization on multiple sub-networks of complex networks

In Sect. 3.1, some cluster synchronization criteria on multiple sub-networks of complex networks have been proposed, and it is known to us that the pinning controllers are applied on just a small fraction of nodes in each followers' sub-network; next, we will make a brief analysis to the condition (19) in Theorem 2. Let $L = \frac{1}{2}\tilde{P} + \frac{\mu}{2}I_N + \tilde{l}\exp(\mu\tau_t)\lambda_{\max}(\tilde{R})I_N$ be a diagonal matrix, $C = \{1, \dots, r_1, r_1 + 1, \dots, r_2, \dots, N\}$ be a node set, which consists of all the followers nodes, $\chi = \{1, \dots, l_1, r_1 + 1, \dots, l_2, \dots, l_m\}$ be a pinned node set, which is composed of all the pinned nodes in the global followers' network, and $\alpha_0 = \max\{\alpha_i^{(\varphi_i)}\}$, $\tilde{\alpha}_0 = \max\{\alpha_j^{(\varphi_j)}\}$, $p_0 = \max\{p_i^{(\varphi_i)}\}$, $\tilde{p}_0 = \max\{p_j^{(\varphi_j)}\}$, where $i \in \chi$ and $j \in C - \chi$. Then, we have

$$\Sigma^* = \psi(\Sigma_1) = \tilde{\Delta}^* + (\tilde{C}G)^* - (\tilde{C}D)^* + L^* = \begin{pmatrix} \hat{A} - \hat{D} & \hat{B} \\ \hat{B}^T & \hat{C} \end{pmatrix} \quad (26)$$

where ψ is a suitable matrix transformation, such that the pinned nodes in a followers' sub-network follow the other pinned nodes from the other ones in sequence, and the superscript "*" represents the corresponding matrix after transformation. As the fact that all the elements of the matrix $\hat{B}$ are positive, it is easy to know that $\begin{pmatrix} \hat{B}^T & \hat{C} \end{pmatrix}$ is a ζ-row-sum matrix when $c_k (k = 1, 2, \ldots, m)$ are assumed to be the same, where $\zeta = \tilde{\alpha}_0 + \frac{\tilde{p}_0}{2} + \frac{\mu}{2} + \tilde{l}\exp(\mu\tau_t)\lambda_{\max}(\tilde{R})$. Therefore, if there are more connections in $\hat{B}$, the matrix $\hat{C}$ will more likely to be a negative definite, which means that the nodes with large degrees should be controlled since these nodes can affect many connected nodes. On the other hand, according to (26) and the fact that the main diagonal elements of a negative definite symmetric matrix are all negative, we can easily get the following inequalities:

$$\begin{cases} \Sigma_{ii}^{*(k)} = \alpha_i^{(k)} + c_k g_{ii}^{(k)} - c_k \sum\limits_{j=w_{k-1}+1}^{w_k} d_{ij}^{(k)} + \frac{1}{2} p_i^{(k)} \\ \quad + \frac{\mu}{2} + \tilde{l}\exp(\mu\tau_t)\lambda_{\max}(\tilde{R}) < 0, \quad i = r_{k-1}+1, \ldots, l_k \\ \Sigma_{ii}^{*(k)} = \alpha_i^{(k)} + c_k g_{ii}^{(k)} + \frac{1}{2} p_i^{(k)} + \frac{\mu}{2} \\ \quad + \tilde{l}\exp(\mu\tau_t)\lambda_{\max}(\tilde{R}) < 0, \quad i = l_k+1, \ldots, r_k \end{cases} \tag{27}$$

In view of $g_{ii}^{(k)} < 0$ and the fact that $\mathrm{Deg}(i)^{(k)} = -g_{ii}^{(k)}$, where $k = 1, \ldots, m$, we can get that the nodes with low degrees should be controlled from (27), which is very consistent with the common intuition that the nodes with very low degrees can receive little information from other nodes.

As for the selection on the number of the pinned nodes, from the above analysis, we can get that all the pinned nodes in each followers' sub-network should satisfy the condition $\zeta I_{[\chi]} + (\tilde{C}G)^*_{[\chi]} < 0$, where $(\tilde{C}G)^*_{[\chi]}$(or $I_{[\chi]}$) represents the minor matrix of $(\tilde{C}G)^*$(or I_N) by removing its first $[\chi]$ row–column pairs, and $[\chi] \in \Re$ denotes the number of elements in the node set χ. Furthermore, as the fact that $(\tilde{C}G)^*_{[\chi]} \leq \lambda_{\max}(\tilde{C}G)^*_{[\chi]} I_{[\chi]}$, we have $\lambda_{\max}[\zeta I_N + (\tilde{C}G)^*]_{[\chi]} \leq \zeta + c_{\max}\lambda_{\max}(G^*_{[\chi]})$. In order to satisfy the condition $[\zeta I_N + (\tilde{C}G)^*]_{[\chi]} < 0$, where $(G)^*_{[\chi]}$ is a symmetric matrix with all the diagonal elements being negative, we only need to check the condition $\zeta + c_{\max}\lambda_{\max}(G^*_{[\chi]}) < 0$ to determine the least number of the pinned nodes in each sub-network. The selection process for the pinned nodes can be simply stated as follows:

(1) Rearrange the nodes according to their degrees from high to low in each followers' sub-network;
(2) According to the condition $\zeta + c_{\max}\lambda_{\max}(G^*_{[\chi]}) < 0$, we can get the least number of the nodes that need to be pinned in each followers' sub-network;
(3) The selection for the pinned nodes is from both the left and the right to the middle in turn at the same time, and the selection of priority for the nodes on the relative position depends on the specific network structure. If the degrees of the nodes are the same, we select these nodes in sequence.

Remark 5 The matrix $\hat{B}$ is positive definite because in our network model, only the pinned nodes can communicate with the other ones belonging to different followers' sub-networks, and the connections between these pinned nodes in different followers' sub-networks are all transferred to the matrix $\hat{A}$ after suitable matrix transformation. According to the definition of the matrix G and the matrix transformation ψ, we can easily get to know that all of the connections between the pinned nodes and the un-pinned nodes are transformed to the matrix $\hat{B}$ or $\hat{B}^T$, and the connections between the un-pinned nodes are all transformed to the matrix $\hat{C}$. Therefore, all of the elements in $\hat{B}$ are nonnegative, and all the non-diagonal elements in $\hat{C}$ are nonnegative but the diagonal elements of $\hat{C}$ are negative.

4 Robustness analysis of the cluster synchronization on multiple sub-networks of complex networks via pinning scheme

As is known to all, most of the previous works on the cluster synchronization with pinning scheme are having all the nodes synchronized to a virtual node in each sub-network, from which all the nodes in the matching sub-network can receive the information. A serious drawback for these schemes is that the network is fragile to the deliberate attacks, where the attack simply means that some nodes or edges are removed from the global network. However, in our proposed scheme, the large-scale complex networks consist of multiple pairs of matching sub-networks, while each pair of matching sub-networks can perform well independently even if there are some connections of the nodes belonging to different sub-networks. Therefore, if the edges or the nodes in a pair of matching sub-networks are removed, the cluster synchronization in the other pairs of matching sub-networks can be proceeded well. As for the attacked sub-network, one only needs to check whether the conditions proposed are satisfied or not. This is superior to the previous literature that, if the sole virtual leader or even one node in the global network is attacked, the whole network can be in a mess and cannot reach synchronization anymore, while this problem can be solved well by the method proposed in this paper.

In general, the nodes in large-scale complex networks can be divided into three categories: the leaders, the pinned nodes and un-pinned node, so the attack can be made on these three kinds of nodes. For the attack on the leaders, this is very important for the current framework. In this improved framework, there are many leaders in each leaders' sub-networks, which means that the proposed scheme in this paper provides more opportunities for the leaders to transmit information. Though the kth sub-network may not reach cluster synchronization if the condition (6) is not satisfied under the attack, the synchronization of the nodes in the other sub-networks can still be reached, which is very superior to the original pinning framework that the synchronization must not be achieved under the attack on the sole virtual leader. Furthermore, it is desirable to design some suitable protocols such that the conditions in Theorem 1 are satisfied if some of the leaders in the kth leaders' sub-network are attacked. For the attack on the pinned nodes and un-pinned nodes, if the pinned or un-pinned nodes in certain followers' sub-networks are attacked and the condition (19) is not satisfied, the nodes in these attacked followers' sub-network may not reach cluster synchronization. However, for the other sub-networks without attacks, the synchronization of the nodes can still be reached, which cannot cause a cascading failure in the global followers' network. In short, the proposed framework for pinning cluster synchronization on multiple sub-networks of complex networks has good robustness to the deliberate attacks, which could be very reasonable and useful for practical applications.

5 Numerical simulations

In this section, some examples are simulated to verify the theoretical analysis proposed in this paper. Consider a general complex network which consists of three pairs of different sub-networks, while the dynamical behavior of the nodes in each pair of matching sub-networks are the same. Three different dynamical systems, such as CNN's neuron system [53], Chua's circuit [54], and Liu-Chen system [55], are selected as the network nodes for these different pairs of matching sub-networks, respectively.

Case 1 *The first pair of matching sub-networks* The first pair of matching sub-networks consists of four followers and three leaders, in which CNN's neuron system [53] is taken as the corresponding node dynamics, and only the first two nodes in the followers' sub-network need to be controlled. The 3D CNN's neuron system that we considered is described by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A_1} \underbrace{\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} p_1 g(x_1) - s g(y_1) - s g(z_1) \\ -s g(x_1) + p_2 g(y_1) - r g(z_1) \\ -s g(x_1) + r g(y_1) + p_3 g(z_1) \end{pmatrix}}_{f_1(x)} \tag{28}$$

where $x = (x_1, y_1, z_1)^T \in \Re^3$, $g(x) = \frac{1}{2}(|x+1| - |x-1|)$, $p_1 = 1.25$, $p_2 = 1.1$, $p_3 = 1$, $s = 3.2$, $r = 4.4$. As indicated by Zou and Nossek [53], system (28) has a double-scrolling chaotic attractor. In view of (A1), (A2) and Lemma 1, we have

$$(x - \bar{s})^T (f_1(x) - f_1(\bar{s})) \le \lambda_{\max}(B_1)(x - \bar{s})^T (x - \bar{s}) \tag{29}$$

where $B_1 = \begin{pmatrix} 1.25 & -3.2 & -3.2 \\ -3.2 & 1.1 & -4.4 \\ -3.2 & 4.4 & 1 \end{pmatrix}$, and

$$\begin{aligned}(x - \bar{s})^T (f_1(t, x, x_{\tau_t}) - f_1(t, \bar{s}, \bar{s}_{\tau_t})) &\le \frac{1}{2}\lambda_{\max}(B_1{}^T B_1)(x - \bar{s})^T (x - \bar{s}) \\ &\quad + \frac{1}{2}\lambda_{\max}(B_1{}^T B_1)(x_{\tau_t} - \bar{s}_{\tau_t})^T (x_{\tau_t} - \bar{s}_{\tau_t})\end{aligned} \tag{30}$$

Case 2 *The second pair of matching sub-networks* The second pair of matching sub-networks consists of five followers and three leaders with Chua's circuit [54] as the node dynamics, and only the first three nodes in the followers' sub-network are in need of being controlled, where the Chua's circuit is described in the form by

$$\begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} a(y_2 - g(x_2)) \\ x_2 - y_2 + z_2 \\ -by_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}}_{A_2} \underbrace{\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} 2x_2 + ay_2 - ag(x_2) \\ x_2 + z_2 \\ -by_2 + 2z_2 \end{pmatrix}}_{f_2(x)} \quad (31)$$

where $x = (x_2, y_2, z_2)^T \in \Re^3$, $g(x) = \frac{2}{7}x - \frac{3}{14}(|x+1| - |x-1|)$, $a = 9$ and $b = \frac{100}{7}$. As shown in [54], the dynamical behavior of the Chua's circuit is chaotic. In view of (A1), (A2) and Lemma 1, we have

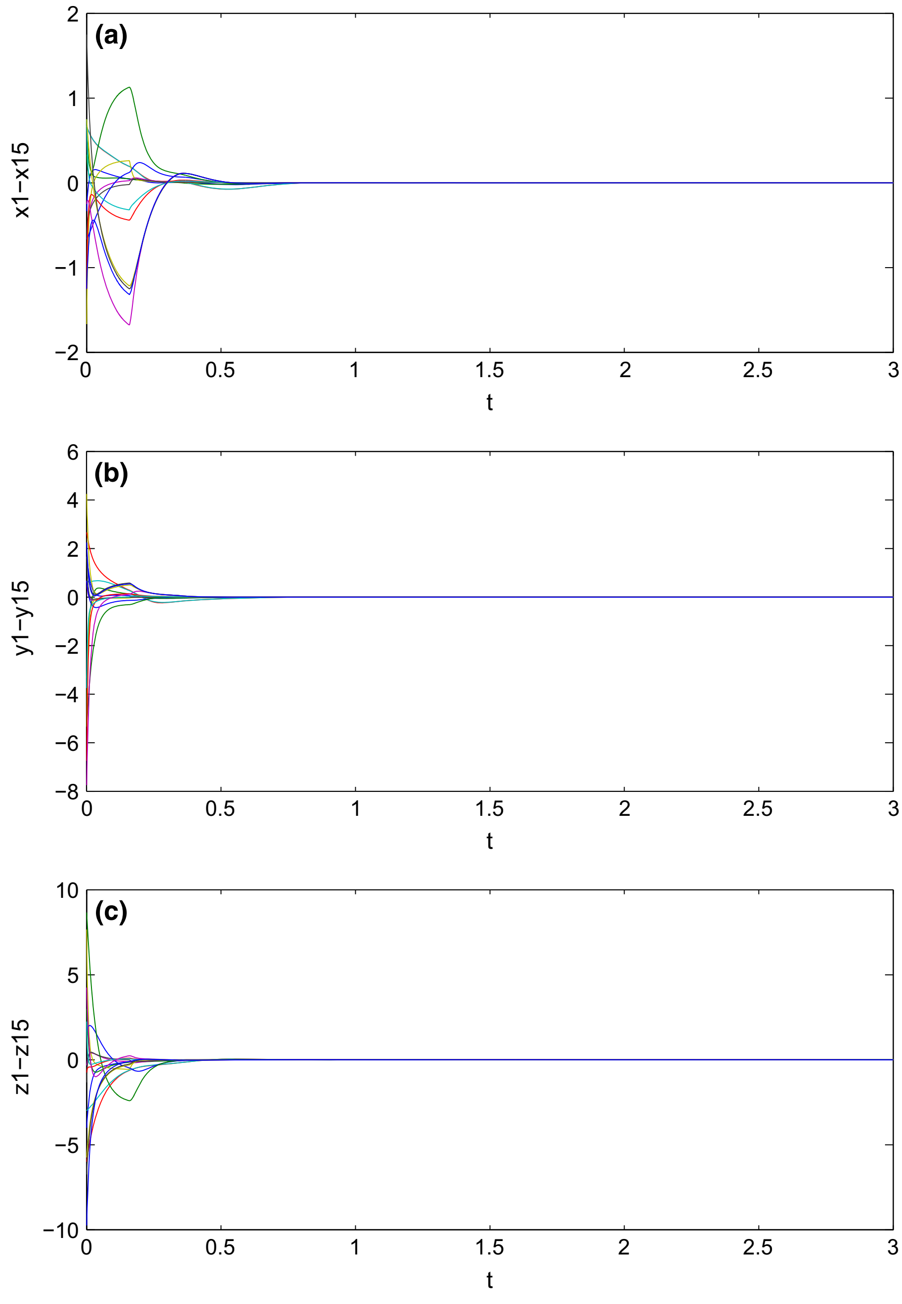

Fig. 2 States of errors between the nodes $x_i^{(\varphi_i)} (i = 1, \ldots, 15)$ and their matching reference states $\bar{s}^{(\varphi_i)}$ in the global followers' network

$$
\begin{aligned}
&(x_i-\bar{s})^T\left[f_2(x_i)-f_2(\bar{s})\right]\\
&\le (x_i-\bar{s})^T B_2(x_i-\bar{s})+\frac{1}{7}a(x_i-\bar{s})^T(x_i-\bar{s})\\
&\le \left(\lambda_{\max}(B_2)+\frac{1}{7}a\right)(x_i-\bar{s})^T(x_i-\bar{s}) \qquad (32)
\end{aligned}
$$

where $B_2=\begin{pmatrix}2 & a & 0\\ 1 & 0 & 1\\ 0 & -b & -2\end{pmatrix}$, and

$$
\begin{aligned}
&(x_i-\bar{s})^T\left[f_2(x_{i\tau_t})-f_2(\bar{s}_{\tau_t})\right]\\
&\le (x_i-\bar{s})^T B_2(x_{i\tau_t}-\bar{s}_{\tau_t})+\frac{1}{7}a(x_i-\bar{s})^T(x_{i\tau_t}-\bar{s}_{\tau_t})
\end{aligned}
$$

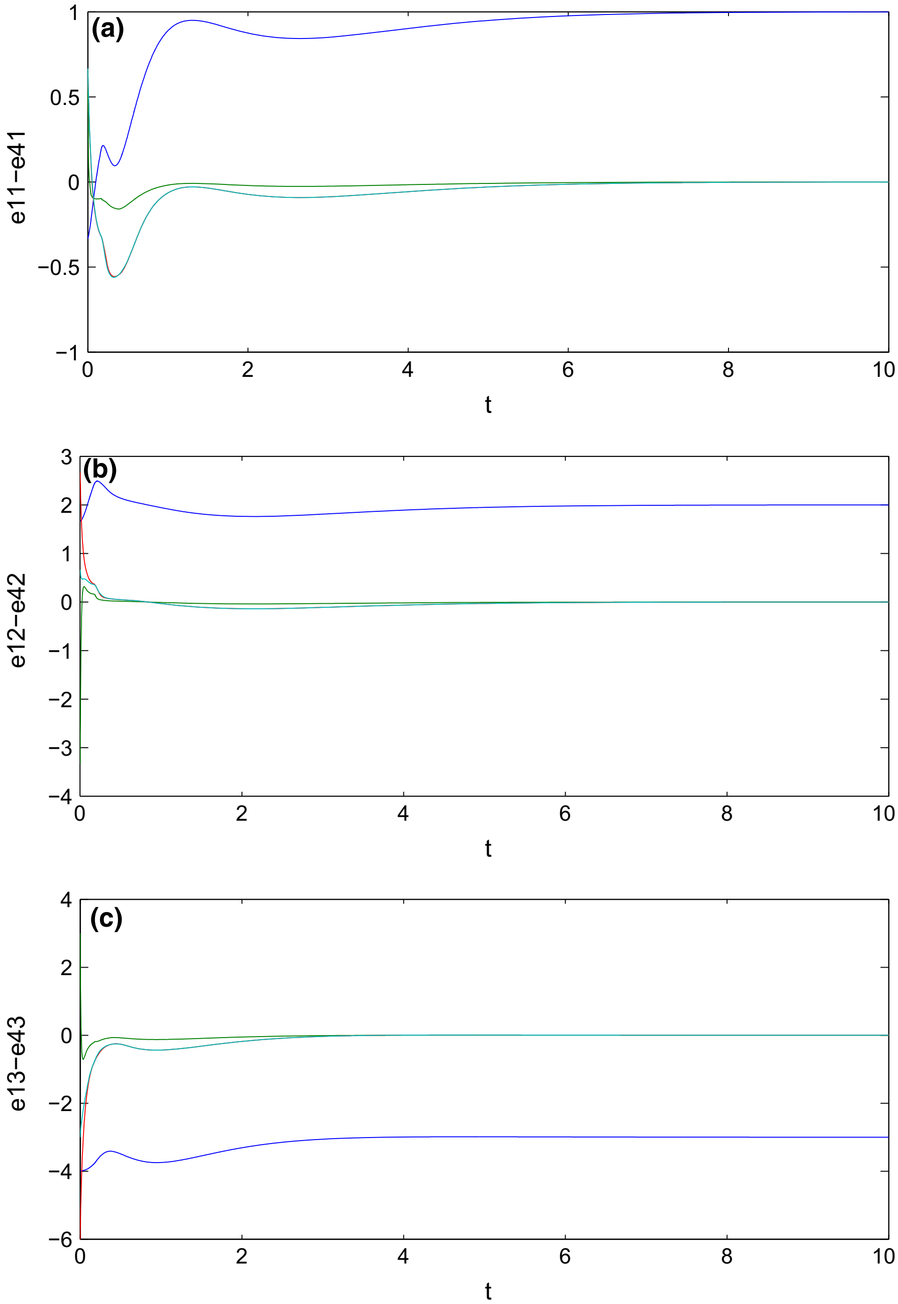

Fig. 3 States of errors between the nodes $x_i^{(1)}(i=1,2,3,4)$ and their matching reference state $\bar{s}^{(1)}$ in the first followers' sub-network with the attack to the first pinned node

$$\leq \frac{p_0}{2}\left(\lambda_{\max}(B_2)+\frac{1}{7}a\right)(x_i-\bar{s})^T(x_i-\bar{s})$$
$$+\frac{1}{2p_0}\left(\lambda_{\max}(B_2)+\frac{1}{7}a\right)(x_{i\tau_t}-\bar{s}_{\tau_t})^T(x_{i\tau_t}-\bar{s}_{\tau_t}),$$
$$p_0>0 \tag{33}$$

Case 3 *The third pair of matching sub-networks* The third pair of matching sub-networks, with Liu-Chen system [55] as the node dynamics, consists of six followers and four leaders, and only the first two nodes in the followers' sub-network need to be controlled. The Liu-Chen system can be described as follows:

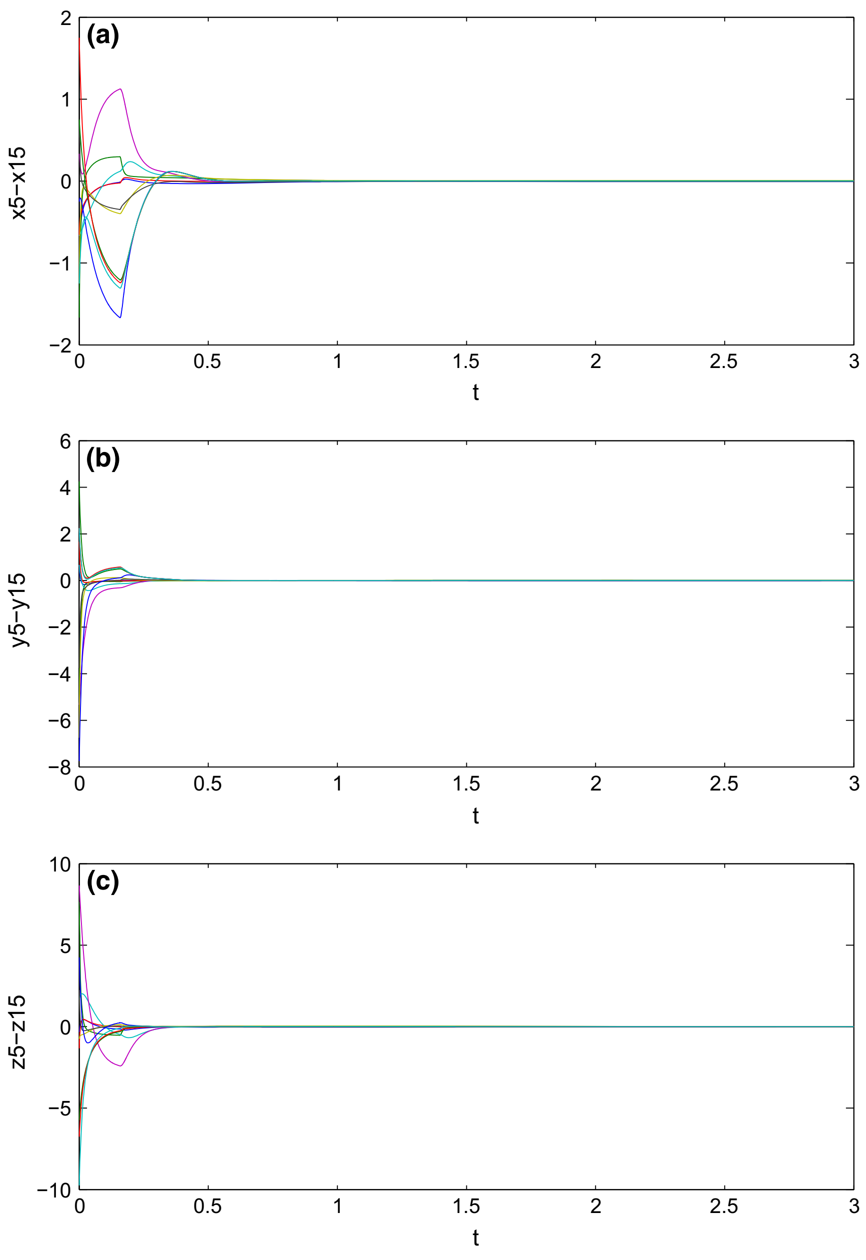

Fig. 4 States of errors between the nodes $x_i^{(\varphi_i)}(i=5,\ldots,15)$ and their matching reference states $\bar{s}^{(\varphi_i)}$ in the second and third followers' sub-network with the attack to the first pinned node in the first followers' sub-network

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -c \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} -y_1 z_1 \\ x_1 z_1 \\ x_1 y_1 \end{pmatrix}$$
$$= \underbrace{\begin{pmatrix} -2 & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -c \end{pmatrix}}_{A_3} \underbrace{\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} (a+2)x_1 - y_1 z_1 \\ x_1 z_1 \\ x_1 y_1 \end{pmatrix}}_{f_3(x)} \tag{34}$$

where $a = 0.4, b = 12, c = 5$, the Liu-Chen system behaves four-scroll chaotic [55]. In view of (A1), (A2) and Lemma 1, we have

$$(x - \bar{s})^T (f_3(x) - f_3(\bar{s})) = 2.4e_1{}^2 + 2Z_1 e_2 e_3$$
$$\leq 2.4e_1{}^2 + \sqrt{Z_1} e_2{}^2 + \sqrt{Z_1} e_3{}^2 = e^T L e \tag{35}$$

where Z_1 is the boundary of the chaotic trajectory on the direction of x, $L = \mathrm{diag}\{2.4, \sqrt{Z_1}, \sqrt{Z_1}\}$, and

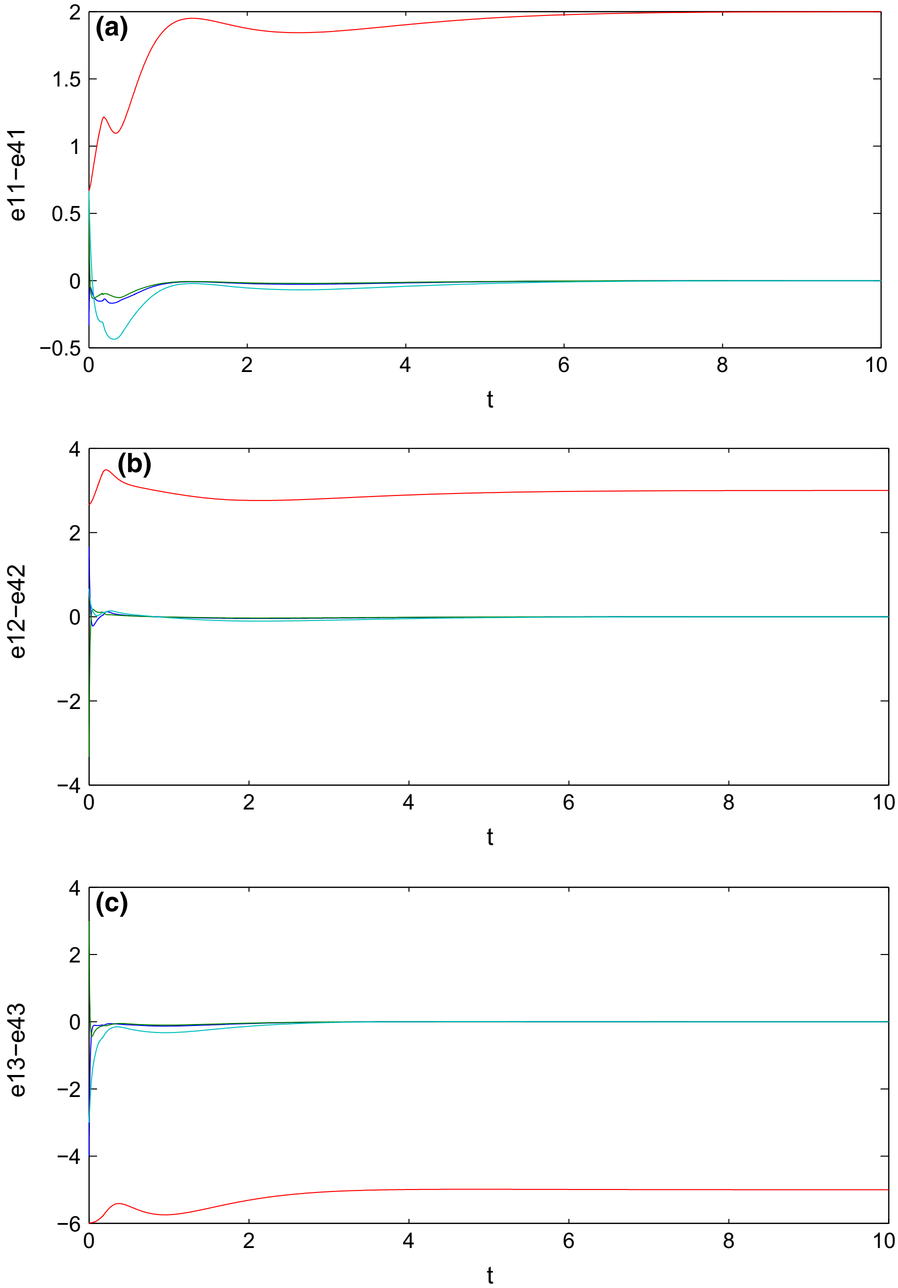

Fig. 5 States of errors between the nodes $x_i^{(1)} (i = 1, 2, 3, 4)$ and their matching reference state $\bar{s}^{(1)}$ in the first followers' sub-network with the attack to the third node (the first un-pinned node)

$$(x_i - \bar{s})^T [f_3(x_{i\tau_t}) - f_3(\bar{s}_{\tau_t})] \le (x_i - \bar{s})^T L(x_{i\tau_t} - \bar{s}_{\tau_t})$$
$$\le \frac{1}{2}\rho(x_i - \bar{s})^T L^T L(x_i - \bar{s})$$
$$+ \frac{1}{2\rho}(x_i - \bar{s}_{\tau_t})^T (x_{i\tau} - \bar{s}_{\tau_t}), \quad \rho > 0 \qquad (36)$$

In the simulation process for the cluster synchronization on three pairs of sub-networks with nonidentical node dynamics, we assumed that $H^{(k)}$, $G_{kk}^{(k)}(k = 1, 2, 3)$ are all fully connected, and the crossover matri-

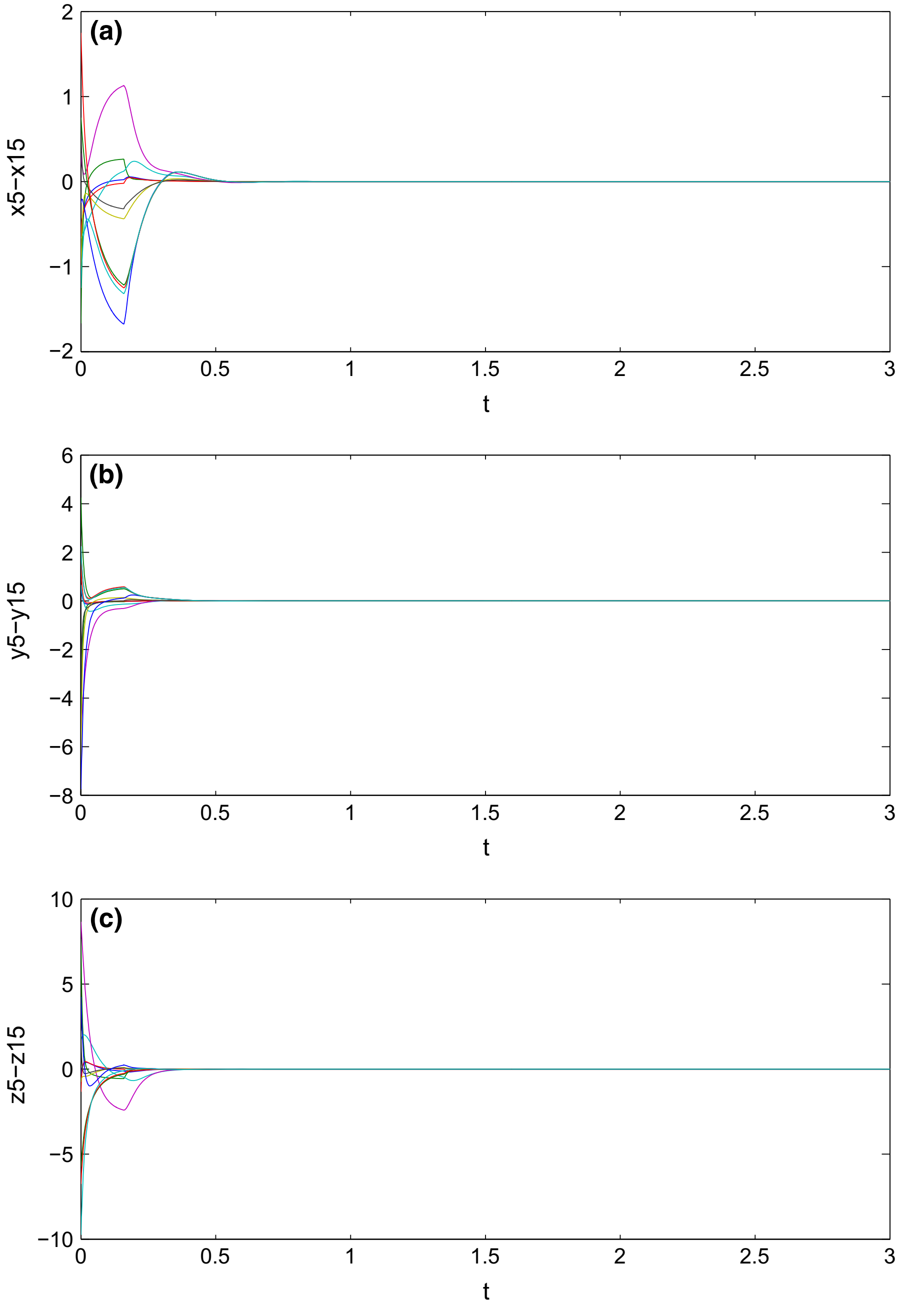

Fig. 6 States of errors between the nodes $x_i^{(\varphi_i)}(i = 5, \ldots, 15)$ and their matching reference states $\bar{s}^{(\varphi_i)}$ in the second and third followers' sub-network with the attack to the first un-pinned node in the first followers' sub-network

ces can be taken as $G_{12}^{(1)} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $G_{21}^{(2)} = G_{12}^{(1)T}$, $G_{13}^{(1)} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $G_{23}^{(2)} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $G_{31}^{(3)} = G_{13}^{(1)T}$, $G_{32}^{(3)} = G_{23}^{(2)T}$. The correspond-ing initial conditions for the other unknown parame-

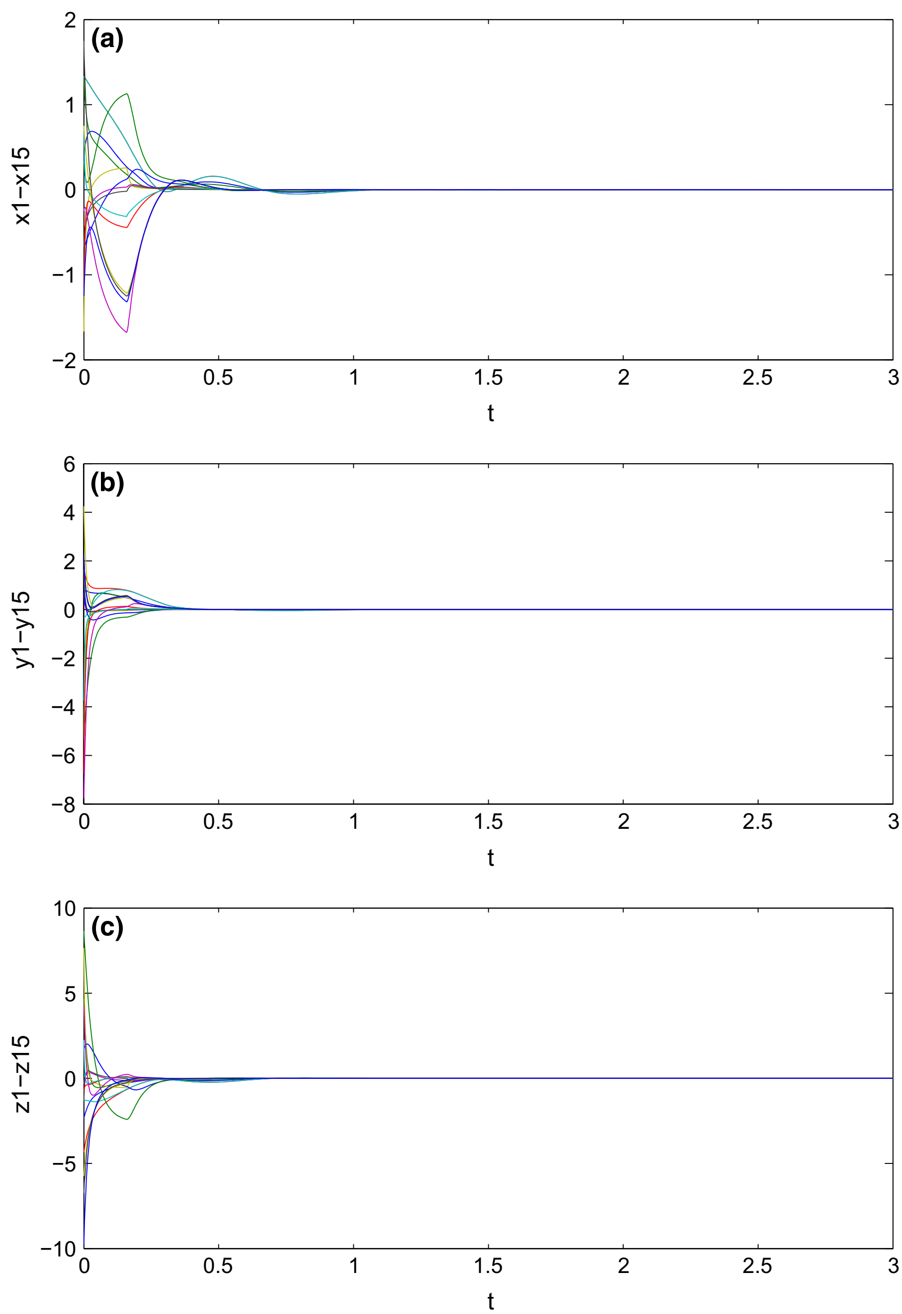

Fig. 7 States of errors between the nodes $x_i^{(\varphi_i)} (i = 1, \ldots, 15)$ and their matching reference states $\bar{s}^{(\varphi_i)}$ in the global followers' network with the attack to the first leader in the first leaders' sub-network

ters are selected as $c_1 = 8$, $\tau_{1t} = e^t/(6+6e^t)$, $d_{11}^{(1)} = 3, d_{12}^{(1)} = 2, d_{13}^{(1)} = 1$, $d_{21}^{(1)} = 2, d_{22}^{(1)} = 3, d_{23}^{(1)} = 4$; $c_2 = 10$, $\tau_{2t} = e^t/(5+5e^t)$, $d_{11}^{(2)} = 3, d_{12}^{(2)} = 4, d_{13}^{(2)} = 5$, $d_{21}^{(2)} = 2, d_{22}^{(2)} = 5, d_{23}^{(2)} = 8$, $d_{31}^{(2)} = 3, d_{32}^{(2)} = 6, d_{33}^{(2)} = 7$; $c_3 = 12$, $\tau_{3t} = e^t/(8+8e^t)$, $d_{11}^{(3)} = 2, d_{12}^{(3)} = 3, d_{13}^{(3)} = 5, d_{14}^{(3)} = 2$, $d_{21}^{(3)} = 3, d_{22}^{(3)} = 4, d_{23}^{(3)} = 6, d_{24}^{(3)} = 5$; $\mu = 1$, $l_1 = l_2 = l_3 = \tilde{l} = 1$, $p_i^{\varphi_i} = q_i^{\varphi_i} = 0.1$, and $\delta(t, x_i, x_{i\tau_t}) = 0.05(x_i - x_{i\tau_t})(i = 1, 2, \ldots, 15)$. The states of the errors between the nodes in the kth followers' sub-network and the average state of their matching leaders $\bar{s}^{(k)}(t)(k = 1, 2, 3)$ are shown in Fig. 2. From the illustration of this figure, one can get that all the errors between the nodes in the global followers' network and their average states of the matching leaders converge to zero very quickly, which is to say that the realization of the cluster synchronization on multiple sub-networks of complex networks has good immunity to the influence of random factors.

On the other hand, cluster synchronization can still be reached even if some attacks occur in the following three types of nodes: (1) Attack a pinned node or an un-pinned node in a followers' sub-network. When the first node (a pinned node) or the third node (an un-pinned node) in the first followers' sub-network is attacked, the other nodes in this sub-network can still realize the cluster synchronization, only the time for the realization of the cluster synchronization becomes a little longer, which are shown in Figs. 3 and 5, respectively. The lines that do not tend to 0 represent that the attacked pinned or un-pinned node does not synchronize to the average state of their matching leaders. However, as for the other followers' sub-networks, there is no obvious influence on the realization of the cluster synchronization, just as shown in Figs. 4 and 6, respectively. From these figures, we can easily get that the attack on a pinned node or an un-pinned node in a followers' sub-network has no large influence on the realization of the cluster synchronization for the other un-attacked nodes. (2) Attack a leader in a leaders' sub-network. When the first leader in the first leaders' sub-network is attacked, the nodes, no matter in the matching or unmatched followers' sub-networks, can still realize the cluster synchronization in a very short time, just as shown in Fig. 7. That is to say, the attack on a leader has no influence on the realization of the cluster synchronization of the nodes in the global followers' network.

From the illustrations of these figures, one can get that the theoretical analysis is consistent with the numerical simulations. The original complex networks are divided into multiple pairs of matching sub-networks, where each pair of matching sub-networks can perform well independently only with the satisfaction of the conditions (6) and (19), even if the attacks occur in some pinned nodes, un-pinned nodes or leaders in complex networks. That is to say, the proposed scheme has good robustness to the deliberate attacks on the cluster synchronization of multiple sub-networks of complex networks. Different from recently literature [42], we not only solve the problem of the cluster synchronization on multiple sub-networks of complex networks with nonidentical node dynamics but also consider the influence of stochastic factors. More importantly, the proposed network structure model is more realistic, and it may have much more practical application in the near future.

6 Conclusions

In this paper, cluster synchronization on multiple sub-networks of complex networks with nonidentical nodes, stochastic disturbances and time-varying delays via pinning control scheme is established, where the complex networks consist of multiple pairs of matching sub-networks. In each pair of matching sub-networks, the nodes can communicate with each other in a sub-network, and only the pinned nodes in the followers' sub-networks can receive information from their leaders' sub-network, but not vice versa. In addition, all of the pinned nodes belonging to different followers' sub-networks can communicate with each other as well. Some cluster synchronization criteria and a pinning control scheme for multiple sub-networks of complex networks are designed, which shows that the nodes with very large or low degrees are good candidates for applying pinning controllers. It should be noted that the proposed scheme under this new framework is testified to have good robustness to the deliberate attacks compared with some traditional pinning schemes applied to the original framework.

The most prominent place in this paper is that we introduce an improved network structure model consisting of multiple pairs of matching sub-networks, where the nodes belonging to different pairs of unmatched sub-networks are assumed to be nonidentical, to realize the cluster synchronization on multiple sub-networks of complex networks; some random factors are taken into consideration as well, which is quite different from the previous pinning schemes and even the pinning scheme proposed in Ref. [42] recently. In fact, there are many leaders in each leaders' sub-network, from which the pinned nodes in the matching followers' sub-network can receive the information, and the average state of all these leaders is regarded as the reference state, which provides a certain ability to resist the deliberate attacks. Apart from this, though the proposed network structure model in this paper is simple, it is much more realistic than the original pinning control model and of great interest to investigate the theoretical analysis and practical applications for the cluster synchronization on multiple sub-networks of complex networks. In the future, we will investigate some other works on the network synchronization of complex networks, such as hybrid control, finite-time control, the discussion on the influence of network structure, and some application of the proposed scheme on all kinds of research fields such as the onset of epilepsy in the brain, multi-agent systems, and Markovian jump stochastic systems.

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References

1. Watts, D.J., Strogatz, S.H.: Collective dynamics of 'small-world' networks. Nature **393**, 440–442 (1998)
2. Barabási, A.L., Albert, R.: Emergence of scaling in random networks. Science **286**, 509–512 (1999)
3. Arenas, A., Díaz-Guilera, A., Kurths, J., Moreno, Y., Zhou, C.: Synchronization in complex networks. Phys. Rep. **469**, 93–153 (2008)
4. Li, Z., Chen, G.: Global synchronization and asymptotic stability of complex dynamical networks. IEEE Trans. Circuits Syst. II **53**, 28–33 (2006)
5. Liu, Y., Wang, Z., Liang, J., Liu, X.: Synchronization and state estimation for discrete-time complex networks with distributed delays. IEEE Trans. Syst. Man Cybern. B **38**, 1314–1325 (2008)
6. Wang, Q., Duan, Z., Perc, M., Chen, G.: Synchronization transitions on small-world neuronal networks: effects of information transmission delay and rewiring probability. EPL **83**, 50008 (2008)
7. Wang, Q., Perc, M., Duan, Z., Chen, G.: Impact of delays and rewiring on the dynamics of small-world neuronal networks with two types of coupling. Phys. A **389**, 3299–3306 (2010)
8. Wang, Q., Perc, M., Duan, Z., Chen, G.: Synchronization transitions on scale-free neuronal networks due to finite information transmission delays. Phys. Rev. E **80**, 026206 (2009)
9. Wang, Q., Chen, G., Perc, M.: Synchronous bursts on scale-free neuronal networks with attractive and repulsive coupling. PLoS One **6**, e15851 (2011)
10. Wang, L., Dai, H., Kong, X., Sun, Y.: Synchronization of uncertain complex dynamical networks via adaptive control. Int. J. Robust Nonlinear Control **19**, 495–511 (2009)
11. Wu, W., Chen, T.: Global synchronization criteria of linearly coupled neural network systems with time-varying coupling. IEEE Trans. Neural Netw. **19**, 319–332 (2008)
12. Wong, W.K., Zhang, W., Tang, Y., Wu, X.: Stochastic synchronization of complex networks with mixed impulses. IEEE Trans. Circuits Syst. **I** (60), 2657–2667 (2013)
13. Zhou, L., Wang, C., He, H., Lin, Y.: Time-controllable combinatorial inner synchronization and outer synchronization of anti-star networks and its application in secure communication. Commun. Nonlinear Sci. Numer. Simul. **22**, 623–640 (2015)
14. Zhou, L., Wang, C., Lin, Y., He, H.: Combinatorial synchronization of complex multiple networks with unknown parameters. Nonlinear Dyn. **79**, 307–324 (2015)
15. Wang, J., Feng, J., Xu, C., Zhao, Y.: Cluster synchronization of nonlinearly-coupled complex networks with nonidentical nodes and asymmetrical coupling matrix. Nonlinear Dyn. **67**, 1635–1646 (2012)
16. Chen, T., Liu, X., Lu, W.: Pinning complex networks by a single controller. IEEE Trans. Circuits Syst. **I** (54), 1317–1326 (2007)
17. Yu, W., Chen, G., Lu, J.: On pinning synchronization of complex dynamical networks. Automatica **45**, 429–435 (2009)
18. Yu, W., Chen, G., Wang, Z., Yang, W.: Distributed consensus filtering in sensor networks. IEEE Trans. Syst. Man Cybern. B **39**, 1568–1577 (2009)
19. Song, Q., Cao, J.: On pinning synchronization of directed and undirected complex dynamical networks. IEEE Trans. Circuits Syst. **I** (57), 672–680 (2010)
20. Li, C., Xu, C., Sun, W., Xu, J., Kurths, J.: Outer synchronization of coupled discrete-time networks. Chaos **19**, 013106 (2009)
21. Sun, W., Li, S.: Generalized outer synchronization between two uncertain dynamical networks. Nonlinear Dyn. **77**, 481–489 (2014)
22. Sun, W., Wang, S., Wang, G., Wu, Y.: Lag synchronization via pinning control between two coupled networks. Nonlinear Dyn. **79**, 2659–2666 (2015)
23. Kaneko, K.: Relevance of dynamic clustering to biological networks. Phys. D **75**, 55–73 (1994)
24. Cao, J., Li, L.: Cluster synchronization in an array of hybrid coupled neural networks with delay. Neural Netw. **22**, 335–342 (2009)
25. Rulkov, N.F.: Images of synchronized chaos: experiments with circuits. Chaos **6**, 262–279 (1996)

26. Kouomou, Y.C., Woafo, P.: Cluster synchronization in coupled chaotic semiconductor lasers and application to switching in chaos-secured communication networks. Opt. Commun. **223**, 283–293 (2003)
27. Wu, W., Zhou, W., Chen, T.: Cluster synchronization of linearly coupled complex networks under pinning control. IEEE Trans. Circuits Syst. **I** (56), 829–839 (2009)
28. Hu, M., Xu, Z., Yang, Y.: Projective cluster synchronization in drive–response dynamical networks. Phys. A **387**, 3759–3768 (2008)
29. Liu, X., Chen, T.: Cluster synchronization in directed networks via intermittent pinning control. IEEE Trans. Neural Netw. **22**, 1009–1020 (2011)
30. Zhang, J., Ma, Z., Chen, G.: Robustness of cluster synchronous patterns in small-world networks with inter-cluster co-competition balance. Chaos **24**, 023111 (2014)
31. Wang, K., Fu, X., Li, K.: Cluster synchronization in community networks with nonidentical nodes. Chaos **19**, 023106 (2009)
32. Wu, J., Jiao, L., Chen, G.: Cluster synchronization in a network of non-identical dynamic systems. Chin. Phys. B **20**, 060503 (2011)
33. Lu, W., Liu, B., Chen, T.: Cluster synchronization in networks of coupled nonidentical dynamical systems. Chaos **20**, 013120 (2010)
34. Hu, A., Cao, J., Hu, M., Guo, L.: Cluster synchronization in directed networks of non-identical systems with noises via random pinning control. Phys. A **395**, 537–548 (2014)
35. Gao, J., Buldyrev, S.V., Stanley, H.E., Havlin, S.: Networks formed from interdependent networks. Nat. Phys. **8**, 40–48 (2012)
36. Gao, J., Buldyrev, S.V., Havlin, S., Stanley, H.E.: Robustness of a network of networks. Phys. Rev. Lett. **107**, 195701 (2011)
37. Wang, Z., Szolnoki, A., Perc, M.: Interdependent network reciprocity in evolutionary games. Sci. Rep. **3**, 1183 (2013)
38. Jiang, L.L., Perc, M.: Spreading of cooperative behaviour across interdependent groups. Sci. Rep. **3**, 2483 (2013)
39. Wang, Z., Szolnoki, A., Perc, M.: Optimal interdependence between networks for the evolution of cooperation. Sci. Rep. **3**, 2470 (2013)
40. Sun, X., Lei, J., Perc, M., Kurths, J., Chen, G.: Burst synchronization transitions in a neuronal network of subnetworks. Chaos **21**, 016110 (2011)
41. Wang, Z., Szolnoki, A., Perc, M.: Self-organization towards optimally interdependent networks by means of coevolution. N. J. Phys. **16**, 033041(2014)
42. Lu, R., Yu, W., Lu, J., Xue, A.: Synchronization on complex networks of networks. IEEE Trans. Neural Netw. Learn. Syst. **25**, 2110–2118 (2014)
43. Yang, X., Cao, J., Lu, J.: Stochastic synchronization of complex networks with nonidentical nodes via hybrid adaptive and impulsive control. IEEE Trans. Circuits Syst. **I** (59), 371–384 (2012)
44. Zhang, Y., Gu, D.W., Xu, S.: Global exponential adaptive synchronization of complex dynamical networks with neutral-type neural network nodes and stochastic disturbances. IEEE Trans. Circuits Syst. **I** (60), 2709–2718 (2013)
45. Wang, W., Li, L., Peng, H., Xiao, J., Yang, Y.: Stochastic synchronization of complex network via a novel adaptive nonlinear controller. Nonlinear Dyn. **76**, 591–598 (2014)
46. Ma, J., Qin, H., Song, X., Chu, R.: Pattern selection in neuronal network driven by electric autapses with diversity in time delays. Int. J. Mod. Phys. B **29**, 1450239 (2015)
47. Qin, H., Ma, J., Jin, W., Wang, C.: Dynamics of electric activities in neuron and neurons of network induced by autapses. Sci. China Technol. Sci. **57**, 936–946 (2014)
48. Qin, H., Wu, Y., Wang, C., Ma, J.: Emitting waves from defects in network with autapses. Commun. Nonlinear Sci. Numer. Simul. **23**, 164–174 (2015)
49. Kolmanovskii, V.B., Myskis, A.D.: Introduction to the Theory and Applications of Functional Differential Equations. Kluwer Academic Publishers, Dordrencht (1999)
50. Wu, J., Hao, L.: Synchronization in complex delayed dynamical networks with nonsymmetric coupling. Phys. A **386**, 513–530 (2007)
51. Horn, R.A., Johnson, C.R.: Matrix Analysis. Cambridge University Press, Cambridge (1985)
52. Khasminskii, R.: Stochastic Stability of Differential Equations. Springer, Berlin (2012)
53. Zou, F., Nossek, J.A.: Bifurcation and chaos in cellular neural networks. IEEE Trans. Circuits Syst. I Fundam. Theory Appl. **40**, 166–173 (1993)
54. Liu, X., Chen, T.: Exponential synchronization of nonlinear coupled dynamical networks with a delayed coupling. Phys. A **381**, 82–92 (2007)
55. El-Dessoky, M.M.: Anti-synchronization of four scroll attractor with fully unknown parameters. Nonlinear Anal. Real World Appl. **11**, 778–783 (2010)