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Combinatorial synchronization of complex multiple networks with unknown parameters

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Abstract In this paper, a combinatorial inner synchronization within a sub-network, which consists of four-wing chaotic system with unknown parameters and external disturbances as node dynamics, and a combinatorial outer synchronization between different sub-networks are investigated. Based on the Lyapunov stability theory, LaSalle's invariance principle, cluster analysis, and pinning control technique, some sufficient conditions, which can ensure not only the combinatorial inner synchronization of the nodes with identical node dynamics in a sub-network, but also the combinatorial outer synchronization of the sink nodes with identical or nonidentical ones between different sub-networks by a suitable switch control scheme, are obtained. By using the pinning control, only the sink node within a sub-network which has direct connections to the sink nodes in other sub-networks needs to be controlled. Finally, some numerical simulations are presented to demonstrate the feasibility and validity of the obtained results by taking the star-like topological structure as an example.

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1 Introduction

Complex networks, which consist of a large set of highly interconnected nodes, have been widely exist in our real world such as World Wide Web, electrical power grids, metabolic networks, biological networks, ecological networks, and social networks [1–5]. In order to describe the real world better, a lot of network models have been introduced, such as weighted networks [6,7], directed networks [8,9], hierarchical networks [10,11], and community networks [6,12,13].

Synchronization, as an important and interesting collective behavior of complex networks, has drawn more and more attention in recent years. Up to now, various types of synchronization methods have been widely studied, such as projective synchronization [14], generalized projective synchronization [15], function projective synchronization [16], modified function projective synchronization [17], combined synchronization [18], and cluster synchronization [19,20]. Nearly all the above studies have focused on the synchronization of only one complex network, which is named inner synchronization. Recently, the synchronization between two coupled networks, which is called outer synchronization, has been extensively investigated, and some interesting and valuable results have been obtained. Wu et al. [21] investigated the generalized outer synchronization between two different complex dynamical networks. Wang et al. discussed the outer synchronization between two time-delay-coupled complex dynamical networks with nonidentical topological structures in [22]. Wang et al. [23] investigated the mixed outer synchronization between two complex networks with the same topological structure and time-varying coupling delay. In Wu and Fu [24], an outer synchronization between drive-response networks with nonidentical topological structure and unknown parameters was achieved. However, these works on the outer synchronization are all limited to the synchronization between two networks, and there are some difficulties to extend these methods to the synchronization between the three or more ones. More recently, a new kind of synchronization pattern-cluster synchronization, has been proposed for the synchronization of multiple sub-networks. It means that all of the different sub-networks, each of which consists of identical dynamical system, achieve synchronization individually within each sub-network, but typically the synchronous states of these sub-networks are mutually different. In fact, many technological, social, and biological networks in our daily life can be divided naturally into multiple sub-networks, and the nodes in the same sub-network often have the same type of system model. Therefore, the study of the synchronization of multiple sub-networks via cluster analysis has become particularly important. Wu and Lu [25] have investigated a cluster synchronization in adaptive complex dynamical networks with nonidentical nodes by using a local control method and a novel adaptive strategy for the coupling strength of the networks. In Wu and Fu [26], a cluster projective synchronization between community networks with nonidentical nodes has been investigated, and some sufficient conditions for the cluster projective synchronization have been derived. Yao et al. [27] have investigated a new cluster projective synchronization scheme in time-varying delay coupled community networks with nonidentical nodes. Wu and Fu [28] have investigated a cluster mixed synchronization of complex networks with nondelayed coupling based on the linear pinning control scheme and adaptive coupling strength method. However, almost all of the above works have focused on only the study of either the inner synchronization within the same network or the outer synchronization between different networks. To the best of our knowledge, there are very few papers having investigated not only the inner synchronization but also the outer synchronization of multiple sub-networks up to now. As in reality, there are still some situations that the nodes not only within the same sub-network but also between different sub-networks need to interchange information with each other in a synchronous way to accomplish a complex multifunctional task collectively, and different proportionality coefficients of the nodes in multiple sub-networks represent different proportions in completing the complex multifunctional task. Such as a synchronized multi-robot system [29–31], in which each robot interacts with others directly or indirectly, can accomplish a complex multifunctional task in the form of collaboration, which is difficult to be achieved by an individual because the collective behavior of all robots offers more flexibility and maneuverability. Therefore, it becomes particularly important and has extremely profound significance if we can put forward a more general method to realize the synchronization of the nodes not only within the same sub-network but also between different sub-networks.

On the other hand, most of the above works need too many controllers, whose numbers are usually the same as the nodes, how to reduce the number of controllers for the synchronization of multiple networks is an extremely important and meaningful task. In addition, most of the above-mentioned works on the synchronization of complex networks are considered to have an accurate system model with ideal condition. In fact, the ideal condition cannot be satisfied, and all the models may also be disturbed by various external factors in many real situations, whereby it is not easy to determine all the system parameters in advance. Therefore, estimation of unknown parameters and consideration of external disturbance in complex networks are very necessary and crucial. Motivated by these discussions, in this paper, a combinatorial inner synchronization and outer synchronization of complex multiple networks with unknown parameters and external disturbances are investigated. Based on the cluster analysis and pinning control, a star-like topological structure that all nodes with identical node dynamics in a sub-network connect with each other in a certain way only leaving a sole sink node as an interaction center to interchange information with other sink nodes in different sub-network is proposed. And only the sink node within a sub-network that has direct connections to the others in different sub-networks needs to be controlled. As the fact that every sub-network can behave independently, in the inner synchronization phase, we only consider the synchronization of the nodes within the same sub-network, and do not consider the connections between different sub-networks, these connections between the nodes in different sub-networks are only considered in the outer synchronization phase. By introducing a switch control scheme to the sink node, we can choose different connection matrices to couple the sink node and the other nodes within the same sub-network and all the sink nodes in different sub-networks, respectively. Some sufficient conditions for the combinatorial inner synchronization in a sub-network with unknown parameters and external disturbances, and the combinatorial outer synchronization between multiple sub-networks with external disturbances are derived. By applying the method to the star-like network, some numerical simulations testify the feasibility and validity of the proposed method. The rest of the paper is organized as follows. The network model description and some preliminaries are introduced in Sect. 2. Based on the Lyapunov stability theory, LaSalle's invariance principle, cluster analysis and pinning control, a combinatorial inner synchronization and outer synchronization of multiple star-like sub-networks are discussed in Sect. 3. Several numerical simulation results are given in Sect. 4. Some conclusions are finally drawn in Sect. 5.

2 Model description and preliminaries

Based on the concept of birds of a feather flock together, the nodes within the same sub-network always have the same system model. Assume that the complex network considered in this paper consists of N nodes with m sub-networks. As the fact that the nodes in each sub-network can perform well alone, we do not consider the connections between different sub-networks for the inner synchronization. Without loss of generality, the node sets of these sub-networks are $V_1 = \{1, 2, \ldots, r_1\}$, $V_2 = \{r_1 + 1, \ldots, r_2\}, \ldots, V_m = \{r_{m-1} + 1, \ldots, r_m\}$, and we have $r_m = N$. Let $v : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, m\}$, if node i belongs to the lth sub-network, then we have $v_i = l$. The lth sub-network consists of $r_l - r_{l-1}$ coupled identical nodes, and we set $r_0 = 0$, with each node being a n-dimensional chaotic system, the corresponding node dynamics with unknown parameters and external disturbances can be described as follows:

$$\begin{aligned}
\dot{x}_i(t) &= f_{v_i}(x_i(t)) + F_{v_i}(x_i(t))\theta_i + \Delta f_i(t, x_i(t)) \\
&\quad + \sum_{j=r_{v_i-1}+1}^{r_{v_i}} c_{i,j} x_j(t),\ i \in V_{v_i} - \tilde{V}_{v_i} \\
\dot{x}_i(t) &= f_{v_i}(x_i(t)) + F_{v_i}(x_i(t))\theta_i + \Delta f_i(t, x_i(t)) \\
&\quad + \sum_{j=r_{v_i-1}+1}^{r_{v_i}} c_{i,j} x_j(t) + u_{v_i}(t),\ i \in \tilde{V}_{v_i} \qquad (1)
\end{aligned}$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)]^T \in R^n$ is the states variable, and the subscript i represents the ith node, $f_{v_i} : R^n \to R^n$ represents a continuous nonlinear function in the v_ith sub-network, $F_{v_i} : R^n \to R^{n\times p}$ is a vector function in the v_ith sub-network, $\theta_i \in R^p$ represents the unknown parameter of node i, $\Delta f_i : R \times R^n \to R^n$ represents the external disturbances of node i, $(c_{i.j})_{(r_{v_i}-r_{v_{i-1}})\times(r_{v_i}-r_{v_{i-1}})}$ is a zero-row-sum connection matrix, which represents the weighted strength and the topological structure of the sub-network, in which $c_{i,j}$ is defined as follows: if there is a direct connection from node i to node $j (i \neq j)$, then $c_{i,j} \neq 0$, otherwise, $c_{i,j} = 0$, and the diagonal elements are given by $c_{i,i} = -\sum_{j=r_{v_i-1}+1, j\neq i}^{r_{v_i}} c_{i,j}$. $u_{v_i} \in R^n$ is a controller to be designed later. V_{v_i} represents the set of nodes in the v_ith sub-network, $\tilde{V}_{v_i}$ denotes the set of nodes in the v_ith sub-network, in which the nodes have direct connections with the other sink nodes in different sub-networks.

Prior to designing the synchronizing controller in the network, some assumptions must be noted as follows:

A1 The unknown uncertainties $\Delta f_i(x_i)$ are all bounded, which means that there are some positive constants k_{v_i}, k and α_i, such that $\left\| \sum_{i\in V_{v_i}-\tilde{V}_{v_i}} \alpha_i \Delta f_i(x_i) - \sum_{i\in \tilde{V}_{v_i}} \alpha_i \Delta f_i(x_i) \right\| \leq k_{v_i}$ for the combinatorial inner synchronization and $\left\| \sum_{i\in \tilde{V}'_{v}} \alpha_i \Delta f_i(x_i) - \sum_{i\in \tilde{V}_m} \alpha_i \Delta f_i(x_i) \right\| \leq k$ for the combinatorial outer synchronization, where $\tilde{V}'_v = \{\tilde{V}_1, \tilde{V}_2, \ldots, \tilde{V}_{m-1}\}$.

A2 The uncertain parameters θ_i are all norm-bounded, such as $\|\theta_i\| \leq \delta_{\theta_i}$, where δ_{θ_i} are known positive constants and $i = r_{l-1} + 1, \ldots, r_l; l = 1, 2, \ldots, m$.

A3 There is a sufficient small positive constant ε, such that $\left\| \theta_i - \hat{\theta}_i \right\| \geq \varepsilon$ $(i = r_{l-1} + 1, \ldots, r_l; l =$

$1, 2, \ldots, m$). Note that $\left\|\theta_i - \hat{\theta}_i\right\| \geq 0$, and ε is a presupposed positive constant that can be chosen arbitrarily small.

Remark 1 The purpose of introducing ε is to avoid the unknown parameters from appearing in controllers and parameters update laws.

Now, we give some definitions of the combinatorial synchronization of multiple sub-networks and a lemma, which will be used later.

Definition 1 A sub-network with $r_l - r_{l-1}$ $(l = 1, 2 \ldots, m)$ nodes is said to be realizing the combinatorial inner synchronization if there are some positive constants $\alpha_j (j = r_{l-1} + 1, \ldots, r_l)$, such that $\lim\limits_{t\to\infty} \left\| \sum_{i\in V_{v_i} - \tilde{V}_{v_i}} \alpha_i x_i - \sum_{i\in\tilde{V}_{v_i}} \alpha_i x_i \right\| = 0$.

Definition 2 A large-scale complex network formed by m sub-networks is said to be realizing the combinatorial outer synchronization if there are some positive constants $\alpha_q (q \in \tilde{V}_v = \{\tilde{V}_1, \tilde{V}_2, \ldots, \tilde{V}_m\})$ such that $\lim\limits_{t\to\infty} \left\| \sum_{i\in\tilde{V}'_v} \alpha_i x_i - \sum_{j\in\tilde{V}_m} \alpha_j x_j \right\| = 0$.

Lemma 1 (Barbalat lemma) [32] *If $w : R_+ \to R_+$ is a uniformly continuous function for $t \geq 0$ and if the limit of the integral $\lim\limits_{t\to\infty} \int_0^t w(\lambda) d\lambda$ exists and is finite, then $\lim\limits_{t\to\infty} w(t) = 0$.*

For the combinatorial inner synchronization within the v_ith sub-network, we can take the error variable as $e_{v_i} = \sum_{i\in V_{v_i} - \tilde{V}_{v_i}} \alpha_i x_i - \sum_{j\in\tilde{V}_{v_i}} \alpha_j x_j$, and then, the corresponding error dynamical system can be described as

$$
\begin{aligned}
\dot{e}_{v_i} = & \sum_{i\in V_{v_i} - \tilde{V}_{v_i}} \alpha_i \Bigg(f_{v_i}(x_i) + F_{v_i}(x_i)\theta_i \\
& + \Delta f_i(t, x_i) + \sum_{j=r_{v_i-1}+1}^{r_{v_i}} c_{i,j} x_j \Bigg) \\
& - \sum_{j\in\tilde{V}_{v_i}} \alpha_j \Bigg(f_{v_i}(x_j) + F_{v_i}(x_j)\theta_j \\
& + \Delta f_j(t, x_j) + \sum_{k=r_{v_i-1}+1}^{r_{v_i}} c_{j,k} x_k + u_{v_i} \Bigg)
\end{aligned} \tag{2}
$$

Thus, our objective is to design a suitable controller u_{v_i} such that the error dynamical system (2) is asymptotically stable, i.e., $\lim\limits_{t\to\infty} e_{v_i} = 0$, $v_i \in \{1, 2, \ldots, m\}$, which implies that all the nodes within the v_ith sub-network have realized the combinatorial inner synchronization.

After all the nodes within the same sub-network have synchronized, all the unknown parameters have been identified exactly; then, we can realize the combinatorial outer synchronization between different sub-networks with exact system model, both the drive and response networks can be described as

$$
\begin{aligned}
\dot{x}_i(t) = & f_{v_i}(x_i(t)) + \Delta f_i(t, x_i(t)) \\
& + \sum_{j\in\tilde{V}_v} \tilde{c}_{i,j} x_j(t), \; i \in \tilde{V}'_v \\
\dot{x}_i(t) = & f_{v_i}(x_i(t)) + \Delta f_i(t, x_i(t)) \\
& + \sum_{j\in\tilde{V}_v} \tilde{c}_{i,j} x_j(t) + \tilde{u}(t), \; i \in \tilde{V}_m
\end{aligned} \tag{3}
$$

where $(\tilde{c}_{i\cdot j})_{m\times m}$ is the zero-row-sum connection matrix representing the weighted strength and the topological structure between different sub-networks, in which $\tilde{c}_{i,j}$ is defined as follows: if there is a direct connection from node i to node j $(i \neq j)$, then $\tilde{c}_{i,j} \neq 0$, otherwise, $\tilde{c}_{i,j} = 0$, and the diagonal elements are given by $\tilde{c}_{i,i} = -\sum_{j\in\tilde{V}_v, j\neq i} c_{i,j}$. $\tilde{u}(t) \in R^n$ is a designed controller for the outer synchronization between different sub-networks. For the combinatorial outer synchronization between multiple sub-networks, we can take the error variable as $e = \sum_{i\in\tilde{V}'_v} \alpha_i x_i - \sum_{l\in\tilde{V}_m} \alpha_l x_l$; the error dynamical system can be written as

$$
\begin{aligned}
\dot{e} = & \sum_{i\in\tilde{V}'_v} \alpha_i \left(f_{v_i}(x_i) + \Delta f_i(t, x_i) + \sum_{j\in\tilde{V}_v} \tilde{c}_{i,j} x_j \right) \\
& - \sum_{l\in\tilde{V}_m} \alpha_l \left(f_{v_l}(x_l) + \Delta f_l(t, x_l) + \sum_{j\in\tilde{V}_v} \tilde{c}_{l,j} x_j + \tilde{u} \right)
\end{aligned} \tag{4}
$$

In order to ensure that each sink node in different sub-network can realize the combinatorial outer synchronization, suitable controller $\tilde{u}$ should be designed to make the error dynamical system (4) is asymptotically stable, i.e., $\lim\limits_{t\to\infty} e(t) = 0$, which means that all the nodes between different sub-networks have realized the combinatorial outer synchronization.

3 Main results for the combinatorial inner synchronization and outer synchronization

In this section, some sufficient criteria for the combinatorial synchronization of multiple networks are proposed based on the pinning control. As the fact that the nodes within the same sub-network have the same system model, they can synchronize to an identical node, which is taken as a sink node by the inner synchronization, so we only have to let the being synchronized sink node as an interaction center to interchange information with other sink nodes in different sub-networks. In fact, we can choose an arbitrary zero-row-sum matrix as a connection matrix for the sub-network with only one sink node in connection with other ones in different sub-networks. For the sake of simplicity, we take the star-like topological structure as the network model in this paper, and every sub-network has $r_l - r_{l-1} (l = 1, 2, \ldots, m)$ nodes. The nodes in their subordinate lth sub-network are marked as $r_{l-1}+1, \ldots, r_l$ in a counterclockwise direction from outside to inside, and the last node in the lth sub-network is remarked r_l as the sink node, which is the only node that has direct connection to the nodes in different sub-networks.

3.1 Combinatorial inner synchronization within the lth star-like sub-network

As the fact that every sub-network can behave independently, in the inner synchronization phase, we only consider the synchronization of the nodes in the same sub-network, and do not consider the connections between different sub-networks, these connections between the sink nodes in different sub-networks are only considered in the outer synchronization phase. In order to realize the combinatorial inner synchronization, we give the following theorem first.

Theorem 1 *For the lth star-like sub-network, if the controller is selected as*

$$
\begin{aligned}
u_l = \frac{1}{\alpha_{r_l}} \Bigg(& \sum_{i=r_{l-1}+1}^{r_l-1} \alpha_i (f_l(x_i) + F_l(x_i)\hat{\theta}_i \\
& + \sum_{q=r_{l-1}+1}^{r_l} c_{i,q} x_q) - \alpha_{r_l} f_l(x_{r_l}) - \alpha_{r_l} F_l(x_{r_l})\hat{\theta}_{r_l} \\
& - \sum_{q=r_{l-1}+1}^{r_l} \alpha_{r_l} c_{r_l,q} x_q + k_l \operatorname{sign}(e_l) \\
& + \sigma_l e_l + \sum_{i=r_{l-1}+1}^{r_l} \frac{2\left(\delta_{\theta_i}^2 + \delta_{\theta_i} \left\|\hat{\theta}_i\right\|\right)}{\varepsilon} \frac{e_l}{\|e_l\|^2} \Bigg)
\end{aligned}
\tag{5}
$$

where $l = 1, 2, \ldots, m$ represents the number of sub-networks, and the corresponding parameter update laws are selected as

$$
\begin{aligned}
\dot{\hat{\theta}}_i &= \gamma_1 \Bigg(\alpha_i F_l{}^T(x_i) e_l \\
&\quad + \frac{\Delta\theta_i - \hat{\theta}_i}{\varepsilon} \Bigg), \quad i = r_{l-1}+1, \ldots, r_l - 1 \\
\dot{\hat{\theta}}_{r_l} &= \gamma_2 \Bigg(-\alpha_{r_l} F_l{}^T(x_{r_l}) e_l \\
&\quad + \frac{\Delta\theta_{r_l} - \hat{\theta}_{r_l}}{\varepsilon} \Bigg), \quad \dot{\sigma}_l = \gamma_3 e_l{}^T e_l,
\end{aligned}
\tag{6}
$$

where $\gamma_1, \gamma_2, \gamma_3 > 0$, $\Delta\theta_i = [\delta_{\theta_{i1}}, \delta_{\theta_{i2}}, \ldots \delta_{\theta_{ip}}]^T \in R^{p\times 1}$, $\delta_{\theta_{ij}}$ is the upper bound of the jth component of the unknown parameters θ_i, and δ_{θ_i} satisfies the condition that $\delta_{\theta_i} \geq \sqrt{\sum_{j=1}^{p} \delta_{\theta_{ij}}^2} (i = r_{l-1}+1, \ldots, r_l)$. Then, the combinatorial inner synchronization of the nodes within the lth star-like sub-network is realized.

Proof For the lth star-like sub-network, we select the Lyapunov function as

$$
\begin{aligned}
V_l = {} & \frac{1}{2} e_l^T e_l + \frac{1}{2\gamma_1} \sum_{i=r_{l-1}+1}^{r_l-1} \left\|\theta_i - \hat{\theta}_i\right\|^2 \\
& + \frac{1}{2\gamma_2} \left\|\theta_{r_l} - \hat{\theta}_{r_l}\right\|^2 + \frac{1}{2\gamma_3} \left\|\sigma_l - \sigma_l^*\right\|^2
\end{aligned}
\tag{7}
$$

where σ_l^* is an arbitrary positive constant to be determined. Taking the derivate of Eq. (7), and substituting Eq. (5) and Eq. (6) into Eq. (7), we can get

$$
\begin{aligned}
\dot{V}_l = e_l{}^T \Bigg(& \sum_{i=r_{l-1}+1}^{r_l-1} \alpha_i (F_l(x_i)(\theta_i - \hat{\theta}_i) + \Delta f_i(t, x_i)) \\
& - \alpha_{r_l} (F_l(x_{r_l})(\theta_{r_l} - \hat{\theta}_{r_l}) + \Delta f_{r_l}(t, x_{r_l})) \\
& - \sum_{i=r_{l-1}+1}^{r_l} \frac{2\left(\delta_{\theta_i}^2 + \delta_{\theta_i} \left\|\hat{\theta}_i\right\|\right)}{\varepsilon} \frac{e_l}{\|e_l\|^2} \Bigg) \\
& - e_l{}^T (k_l \operatorname{sign}(e_l) + \sigma_l e_l) \\
& + \frac{1}{\gamma_1} \sum_{i=r_{l-1}+1}^{r_l-1} (\theta_i - \hat{\theta}_i)^T
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(-\gamma_1\alpha_i F_l{}^T(x_i)e_l-\gamma_1\frac{\Delta\theta_i-\hat{\theta}_i}{\varepsilon}\right)\\
&+\frac{1}{\gamma_2}(\theta_{r_l}-\hat{\theta}_{r_l})^T\left(\gamma_2\alpha_{r_l}F_l{}^T(x_{r_l})e_l\right.\\
&\left.-\gamma_2\frac{\Delta\theta_{r_l}-\hat{\theta}_{r_l}}{\varepsilon}\right)+\frac{1}{\gamma_3}(\sigma_l-\sigma_l^*)^T\gamma_3 e_l{}^T e_l\\
=\ & e_l{}^T\left(\sum_{i=r_{l-1}+1}^{r_l-1}\alpha_i\Delta f_i(t,x_i)-\alpha_{r_l}\Delta f_{r_l}(t,x_{r_l})\right)\\
&-e_l{}^T\left(k_l\operatorname{sign}(e_l)\right.\\
&\left.+\sigma_l e_l+\sum_{i=r_{l-1}+1}^{r_l}\frac{2\left(\delta_{\theta_i}^2+\delta_{\theta_i}\left\|\hat{\theta}_i\right\|\right)}{\varepsilon}\frac{e_l}{\|e_l\|^2}\right)\\
&-\sum_{i=r_{l-1}+1}^{r_l}(\theta_i-\hat{\theta}_i)^T\frac{\Delta\theta_i-\hat{\theta}_i}{\varepsilon}+(\sigma_l-\sigma_l^*)^T e_l{}^T e_l\\
=\ & e_l{}^T\left(\sum_{i=r_{l-1}+1}^{r_l-1}\alpha_i\Delta f_i(t,x_i)-\alpha_{r_l}\Delta f_{r_l}(t,x_{r_l})\right)\\
&-e_l{}^T\left(k_l\operatorname{sign}(e_l)+\sigma_l e_l\right.\\
&\left.+\sum_{i=r_{l-1}+1}^{r_l}\frac{2\left(\delta_{\theta_i}^2+\delta_{\theta_i}\left\|\hat{\theta}_i\right\|\right)}{\varepsilon}\frac{e_l}{\|e_l\|^2}\right)\\
&-\sum_{i=r_{l-1}+1}^{r_l}\frac{\left\|\theta_i-\hat{\theta}_i\right\|^2}{\varepsilon}\\
&+\sum_{i=r_{l-1}+1}^{r_l}\frac{(\theta_i-\hat{\theta}_i)^T(\theta_i-\Delta\theta_i)}{\varepsilon}\\
&+(\sigma_l-\sigma_l^*)^T e_l{}^T e_l
\end{aligned}\tag{8}
$$

According to **A2**, we can easily get the following inequalities:

$$
\begin{aligned}
&(\theta_i-\hat{\theta}_i)^T(\theta_i-\Delta\theta_i)\\
&\quad=\theta_i{}^T\theta_i-\theta_i{}^T\Delta\theta_i-\hat{\theta}_i^T\theta_i+\hat{\theta}_i^T\Delta\theta_i\\
&\quad\le 2\left(\delta_{\theta_i}^2+\delta_{\theta_i}\left\|\hat{\theta}_i\right\|\right),\ i=r_{l-1}+1,\ldots,r_l
\end{aligned}\tag{9}
$$

In view of **A1** and the inequalities (9), one obtains

$$
\begin{aligned}
\dot{V}_l\le\ & -e_l{}^T\sigma_l e_l-\sum_{i=r_{l-1}+1}^{r_l}\frac{\left\|\theta_i-\hat{\theta}_i\right\|^2}{\varepsilon}\\
&+(\sigma_l-\sigma_l^*)^T e_l{}^T e_l\\
\le\ & -\sum_{i=r_{l-1}+1}^{r_l}\frac{\left\|\theta_i-\hat{\theta}_i\right\|^2}{\varepsilon}-\sigma_l^* e_l{}^T e_l<0
\end{aligned}\tag{10}
$$

Thus, from the inequalities (10), we have $\dot{V}_l<0$. According to the Barbalat lemma and LaSalle's invariance principle, we can easily get the largest invariant set $E=\{e_l\to 0,\theta_i\to\hat{\theta}_i,\sigma_l\to\sigma_l^*,l=1,2,\ldots,m,i=r_{l-1}+1,\ldots,r_l\}$ as $t\to\infty$. □

3.2 Combinatorial outer synchronization between multiple star-like sub-networks

As all the unknown parameters have been exactly identified with the inner synchronization, all of the nodes in the same or different sub-networks have exact system models at present, but these systems may also be disturbed by some external factors, in order to realize the combinatorial outer synchronization between different star-like sub-networks with external disturbances, a theorem is given at first.

Theorem 2 *For multiple star-like sub-networks, the controller can be designed as*

$$
\begin{aligned}
\tilde{u}=\frac{1}{\alpha_{r_m}}&\left(\sum_{i\in Z}\alpha_i\left(f_{v_i}(x_i)+\sum_{j\in\tilde{V}_v}c_{i,j}x_j\right)\right.\\
&-\alpha_{r_m}f_m(x_{r_m})\\
&\left.-\alpha_{r_m}\sum_{j\in\tilde{V}_v}c_{r_m,j}x_j+k\operatorname{sign}(e)+\sigma e\right)
\end{aligned}\tag{11}
$$

where $Z=\{r_1,r_2,\ldots,r_{m-1}\}$, and the corresponding parameter update law can be selected as

$$\dot{\sigma}=\gamma_4 e^T e\tag{12}$$

where $\gamma_4>0$, then the combinatorial outer synchronization of all the sink nodes between multiple star-like sub-networks is realized. The proof is similar to the theorem 1, which is omitted here.

4 Numerical simulations for the combinatorial synchronization of multiple sub-networks

In this section, we will give several numerical examples to demonstrate the effectiveness of the theoretic results for the combinatorial inner synchronization within the same sub-network and outer synchronization between different sub-networks. As the fact that not all the nodes

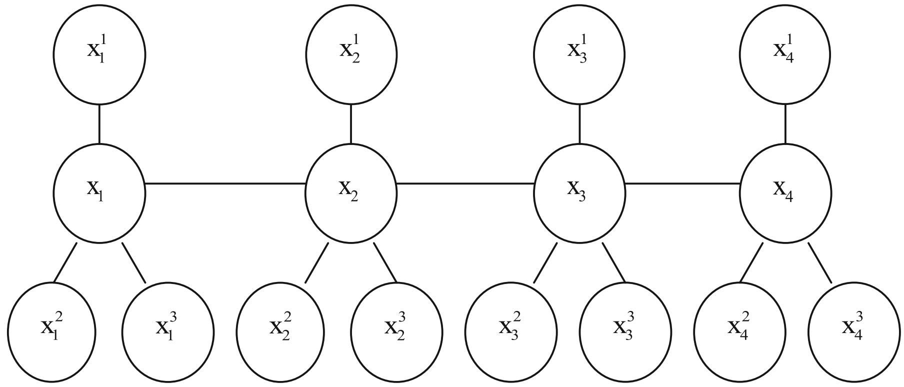

Fig. 1 The topological structure of multiple star-like sub-networks

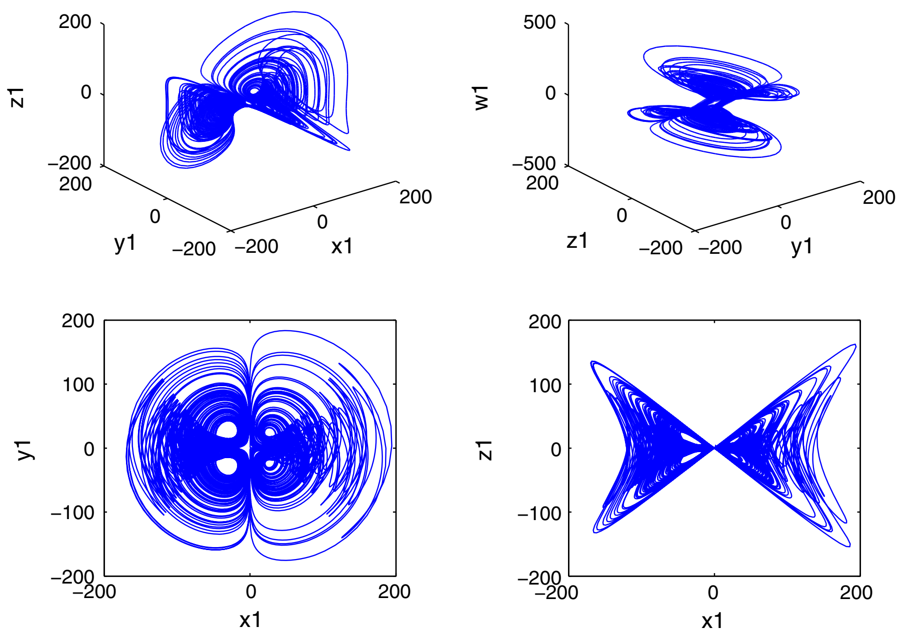

Fig. 2 The phase diagram of the four-wing Yu system

are identical since some complex networks may consist of different types of nodes [33] in reality, such as a synchronized multi-robot system, in which each robot with different node dynamics is assigned to accomplish a part of the complex task with each other synergistically, and different weighting coefficients of the robot represent different proportions in completing the complex task, which leads to the combinatorial synchronization of multiple sub-networks with identical or nonidentical node dynamics. In the following simulations, star-like topological structure is taken as an example, and we may as well set the network size as $N = 16$. Suppose the complex network consists of four sub-networks with identical or nonidentical node dynamics, and every sub-network has four nodes. The topological structure is shown in Fig. 1.

4.1 Combinatorial inner synchronization in a star-like sub-network with unknown parameters and external disturbances

In this paper, for convenience, we only take two different types of four-wing chaotic systems as the local node dynamics for different star-like sub-networks, which are proposed by Yu et al. in [34] and Li et al. in [35], respectively. The Yu system can be described as follows

$$\begin{cases} \dot{x} = yz - ax \\ \dot{y} = by - xz \\ \dot{z} = xy - cz + dw \\ \dot{w} = xz - fw \end{cases} \tag{13}$$

where $a = 10, b = 12, c = 50, d = 2, f = 4$, the four-wing chaotic attractor can be shown as in Fig. 2.

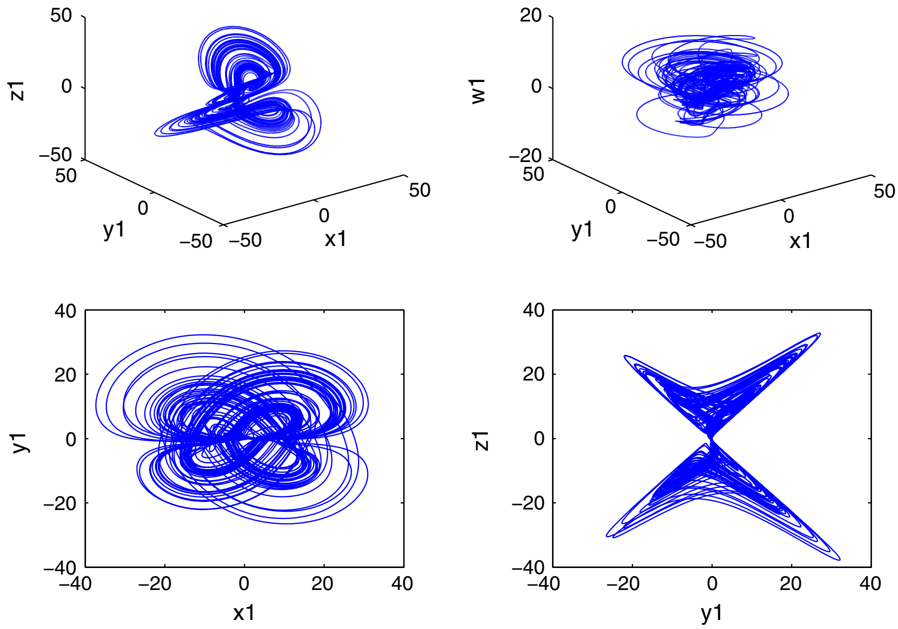

Fig. 3 The phase diagram of the four-wing Li systems

The other four-wing hyper-chaotic Li system can be described as:

$$\begin{cases} \dot{x} = ax - yz + fw \\ \dot{y} = -by + xz \\ \dot{z} = -cz + xy + dx \\ \dot{w} = -mx \end{cases} \tag{14}$$

where $a = 4, b = 12, c = 5.5, d = 1, f = 2.5, m = 1$, the dynamical behavior of the system (14) is hyper-chaotic, and its phase diagram is shown in Fig. 3.

For the combinatorial inner synchronization within the same sub-network, we label the nodes from outside to inside, and all the nodes within a sub-network have the same node dynamics; the star-like connection matrix can be chosen as

$$C = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \tag{15}$$

For the Yu system, the combined driver systems and response system can be described as

$$\dot{x}^1 = \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \\ \dot{w}_1 \end{pmatrix} = \underbrace{\begin{pmatrix} y_1 z_1 - a x_1 \\ -x_1 z_1 \\ x_1 y_1 + d w_1 \\ x_1 z_1 - f w_1 \end{pmatrix}}_{f_1(x^1)} + \underbrace{\begin{pmatrix} 0 & 0 \\ y_1 & 0 \\ 0 & -z_1 \\ 0 & 0 \end{pmatrix}}_{F_1(x^1)} \underbrace{\begin{pmatrix} b_1 \\ c_1 \end{pmatrix}}_{\theta_1}$$

$$+ \underbrace{\begin{pmatrix} 0.1\sin(x_1) \\ 0.2\sin(y_1) \\ 0.3\sin(z_1) \\ 0.4\sin(w_1) \end{pmatrix}}_{\Delta f_1(x^1)} + \underbrace{\begin{pmatrix} x_1 - x_4 \\ y_1 - y_4 \\ z_1 - z_4 \\ w_1 - w_4 \end{pmatrix}}_{\sum_{q=1}^4 c_{1,q} x_q}$$

$$\dot{x}^2 = \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \\ \dot{z}_2 \\ \dot{w}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} y_2 z_2 - a x_2 \\ -x_2 z_2 \\ x_2 y_2 + d w_2 \\ x_2 z_2 - f w_2 \end{pmatrix}}_{f_2(x^2)} + \underbrace{\begin{pmatrix} 0 & 0 \\ y_2 & 0 \\ 0 & -z_2 \\ 0 & 0 \end{pmatrix}}_{F_2(x^2)} \underbrace{\begin{pmatrix} b_2 \\ c_2 \end{pmatrix}}_{\theta_2}$$

$$+ \underbrace{\begin{pmatrix} 0.1\cos(\pi x_2) \\ -0.2\cos(\pi y_2) \\ 0.3\cos(\pi z_2) \\ -0.4\cos(\pi w_2) \end{pmatrix}}_{\Delta f_2(x^2)} + \underbrace{\begin{pmatrix} x_2 - x_4 \\ y_2 - y_4 \\ z_2 - z_4 \\ w_2 - w_4 \end{pmatrix}}_{\sum_{q=1}^4 c_{2,q} x_q}$$

$$\dot{x}^3 = \begin{pmatrix} \dot{x}_3 \\ \dot{y}_3 \\ \dot{z}_3 \\ \dot{w}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} y_3 z_3 - a x_3 \\ -x_3 z_3 \\ x_3 y_3 + d w_3 \\ x_3 z_3 - f w_3 \end{pmatrix}}_{f_3(x^3)} + \underbrace{\begin{pmatrix} 0 & 0 \\ y_3 & 0 \\ 0 & -z_3 \\ 0 & 0 \end{pmatrix}}_{F_3(x^3)} \underbrace{\begin{pmatrix} b_3 \\ c_3 \end{pmatrix}}_{\theta_3}$$

$$+ \underbrace{\begin{pmatrix} 0.1\sin(2\pi x_3) \\ -0.2\cos(2\pi y_3) \\ 0.3\cos(2\pi z_3) \\ -0.4\sin(2\pi w_3) \end{pmatrix}}_{\Delta f_3(x^3)} + \underbrace{\begin{pmatrix} x_3 - x_4 \\ y_3 - y_4 \\ z_3 - z_4 \\ w_3 - w_4 \end{pmatrix}}_{\sum_{q=1}^4 c_{3,q} x_q} \tag{16}$$

$$\dot{x}^4=\begin{pmatrix}\dot{x}_4\\ \dot{y}_4\\ \dot{z}_4\\ \dot{w}_4\end{pmatrix}=\underbrace{\begin{pmatrix}y_4z_4-ax_4\\ -x_4z_4\\ x_4y_4+dw_4\\ x_4z_4-fw_4\end{pmatrix}}_{f_4(x^4)}+\underbrace{\begin{pmatrix}0&0\\ y_4&0\\ 0&-z_4\\ 0&0\end{pmatrix}}_{F_4(x^4)}\underbrace{\begin{pmatrix}b_4\\ c_4\end{pmatrix}}_{\theta_4}$$
$$+\underbrace{\begin{pmatrix}0.1\cos(3\pi x_4)\\ 0.2\sin(3\pi y_4)\\ -0.3\sin(3\pi z_4)\\ 0.4\cos(3\pi w_4)\end{pmatrix}}_{\Delta f_4(x^4)}$$
$$+\underbrace{\begin{pmatrix}-x_1-x_2-x_3+3x_4\\ -y_1-y_2-y_3+3y_4\\ -z_1-z_2-z_3+3z_4\\ -w_1-w_2-w_3+3w_4\end{pmatrix}}_{\sum_{q=1}^{4}c_{4,q}x_q}+\underbrace{\begin{pmatrix}u_1\\ u_2\\ u_3\\ u_4\end{pmatrix}}_{u} \tag{17}$$

where the superscript i represents the ith node, the parameters are selected as $a=10, d=2, f=4$, and $b_i, c_i (i=1,2,3,4)$ are unknown. The error system can be described

$$e=Ax^1+Bx^2+Cx^3-Dx^4$$
$$=\begin{pmatrix}\alpha_1x_1+\beta_1x_2+\gamma_1x_3-m_1x_4\\ \alpha_2y_1+\beta_2y_2+\gamma_2y_3-m_2y_4\\ \alpha_3z_1+\beta_3z_2+\gamma_3z_3-m_3z_4\\ \alpha_4w_1+\beta_4w_2+\gamma_4w_3-m_4w_4\end{pmatrix} \tag{18}$$

where $A=\mathrm{diag}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, $B=\mathrm{diag}(\beta_1,\beta_2,\beta_3,\beta_4)$, $C=\mathrm{diag}(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$, $D=\mathrm{diag}(m_1,m_2,m_3,m_4)$, and according to the detailed theory analysis presented in Sect. 3, we can easily realize the combinatorial inner synchronization within a sub-network. In the simulation process, we take the matrices as $A=\mathrm{diag}(3,2,1,4)$, $B=\mathrm{diag}(2,4,3,2)$, $C=\mathrm{diag}(3,2,4,3)$, $D=\mathrm{diag}(2,3,2,4)$. The choice of initial conditions for the drive systems and response system is arbitrarily, which can be taken as $(x_{10}, y_{10}, z_{10}, w_{10})=(3,-5,6,8)$, $(x_{20}, y_{20}, z_{20}, w_{20})=(5,7,-2,9)$, $(x_{30}, y_{30}, z_{30}, w_{30})=(6,-4,8,10)$ and $(x_{40}, y_{40}, z_{40}, w_{40})=(5,6,10,-6)$. The initial values of estimated parameters are chosen as $(\hat{b}_1,\hat{c}_1)=(1,1)$, $(\hat{b}_2,\hat{c}_2)=(1,1)$, $(\hat{b}_3,\hat{c}_3)=(1,1)$ and $(\hat{b}_4,\hat{c}_4)=(1,1)$. Meanwhile, we assume $k_1=2, k_2=3, k_3=2, k_4=3, \gamma_1=1, \gamma_2=2, \gamma_3=0.01$, $\varepsilon=0.1$, and the initial values of the feedback factors are selected as $\delta_{10}=2, \delta_{20}=2, \delta_{30}=2, \delta_{40}=2$. The time responses of error variables and adaptive parameters $\hat{b}_1,\hat{c}_1,\hat{b}_2,\hat{c}_2,\hat{b}_3,\hat{c}_3,\hat{b}_4,\hat{c}_4$ are shown in Figs. 4, 5 and 6, respectively. It can be seen that the synchronization errors e_1, e_2, e_3, e_4 converge to zero very fast, which means that the combinatorial inner synchronization of Yu system with unknown parameters and external disturbances in the same star-like sub-network is realized. Furthermore, the unknown parameters $b_i, c_i (i=1,2,3,4)$ all tend to the expected values, respectively.

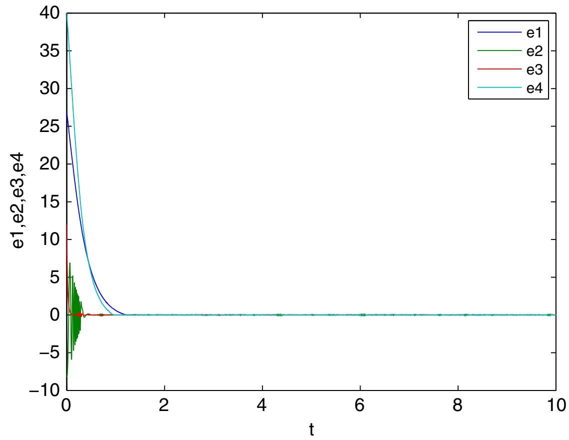

Fig. 4 Time responses of the error variables in Yu system

For the Li system, the combined driver systems and the corresponding response system can be described as

$$\dot{x}^1=\begin{pmatrix}\dot{x}_1\\ \dot{y}_1\\ \dot{z}_1\\ \dot{w}_1\end{pmatrix}=\underbrace{\begin{pmatrix}ax_1-y_1z_1\\ -by_1+x_1z_1\\ -cz_1+x_1y_1\\ -mx_1\end{pmatrix}}_{f_1(x^1)}+\underbrace{\begin{pmatrix}w_1&0\\ 0&0\\ 0&x_1\\ 0&0\end{pmatrix}}_{F_1(x^1)}\underbrace{\begin{pmatrix}f_1\\ d_1\end{pmatrix}}_{\theta_1}$$
$$+\underbrace{\begin{pmatrix}0.1\sin(x_1)\\ 0.2\sin(y_1)\\ 0.3\sin(z_1)\\ 0.4\sin(w_1)\end{pmatrix}}_{\Delta f_1(x^1)}+\underbrace{\begin{pmatrix}x_1-x_4\\ y_1-y_4\\ z_1-z_4\\ w_1-w_4\end{pmatrix}}_{\sum_{j=1}^{4}c_{1,j}x^j}$$

$$\dot{x}^2=\begin{pmatrix}\dot{x}_2\\ \dot{y}_2\\ \dot{z}_2\\ \dot{w}_2\end{pmatrix}=\underbrace{\begin{pmatrix}ax_2-y_2z_2\\ -by_2+x_2z_2\\ -cz_2+x_2y_2\\ -mx_2\end{pmatrix}}_{f_2(x^2)}+\underbrace{\begin{pmatrix}w_2&0\\ 0&0\\ 0&x_2\\ 0&0\end{pmatrix}}_{F_2(x^2)}\underbrace{\begin{pmatrix}f_2\\ d_2\end{pmatrix}}_{\theta_2}$$
$$+\underbrace{\begin{pmatrix}-0.1\cos(x_2)\\ -0.2\cos(y_2)\\ -0.3\cos(z_2)\\ -0.4\cos(w_2)\end{pmatrix}}_{\Delta f_2(x^2)}+\underbrace{\begin{pmatrix}x_2-x_4\\ y_2-y_4\\ z_2-z_4\\ w_2-w_4\end{pmatrix}}_{\sum_{j=1}^{4}c_{2,j}x^j}$$

$$\dot{x}^3=\begin{pmatrix}\dot{x}_3\\ \dot{y}_3\\ \dot{z}_3\\ \dot{w}_3\end{pmatrix}=\underbrace{\begin{pmatrix}ax_3-y_3z_3\\ -by_3+x_3z_3\\ -cz_3+x_3y_3\\ -mx_3\end{pmatrix}}_{f_3(x^3)}+\underbrace{\begin{pmatrix}w_3&0\\ 0&0\\ 0&x_3\\ 0&0\end{pmatrix}}_{F_3(x^3)}\underbrace{\begin{pmatrix}f_3\\ d_3\end{pmatrix}}_{\theta_3}$$

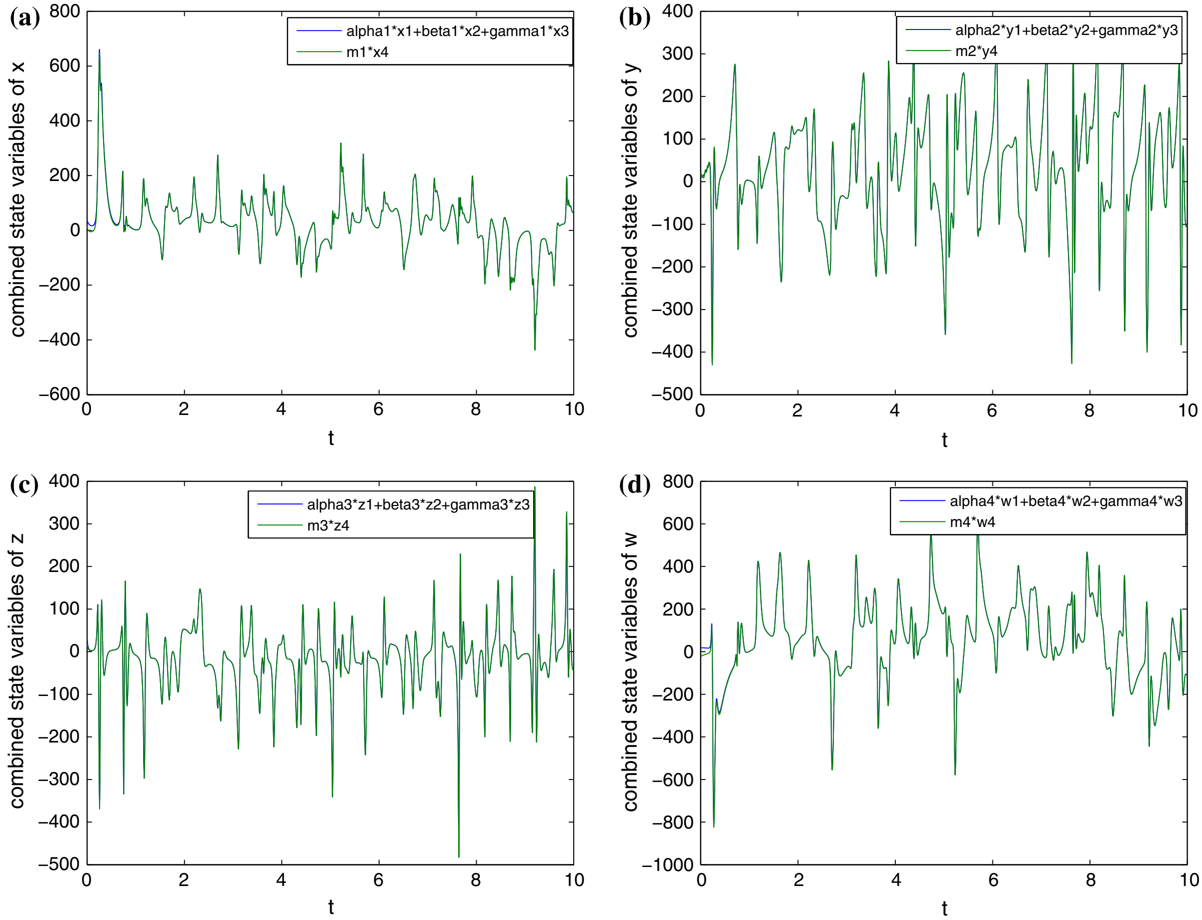

Fig. 5 Time response of the combined variables of Yu system

$$+\underbrace{\begin{pmatrix} 0.1\sin(x_3) \\ -0.2\cos(y_3) \\ 0.3\cos(z_3) \\ -0.4\sin(w_3) \end{pmatrix}}_{\Delta f_3(x^3)} + \underbrace{\begin{pmatrix} x_3 - x_4 \\ y_3 - y_4 \\ z_3 - z_4 \\ w_3 - w_4 \end{pmatrix}}_{\sum_{j=1}^{4} c_{3,j} x^j} \tag{19}$$

$$\dot{x}^4 = \begin{pmatrix} \dot{x}_4 \\ \dot{y}_4 \\ \dot{z}_4 \\ \dot{w}_4 \end{pmatrix} = \underbrace{\begin{pmatrix} ax_4 - y_4 z_4 \\ -by_4 + x_4 z_4 \\ -cz_4 + x_4 y_4 \\ -mx_4 \end{pmatrix}}_{f_4(x^4)} + \underbrace{\begin{pmatrix} w_4 & 0 \\ 0 & 0 \\ 0 & x_4 \\ 0 & 0 \end{pmatrix}}_{F_4(x^4)} \underbrace{\begin{pmatrix} f_4 \\ d_4 \end{pmatrix}}_{\theta_4}$$

$$+\underbrace{\begin{pmatrix} -0.1\cos(x_4) \\ 0.2\sin(y_4) \\ 0.3\cos(z_4) \\ -0.4\sin(w_4) \end{pmatrix}}_{\Delta f_4(x^4)}$$

$$+\underbrace{\begin{pmatrix} -x_1 - x_2 - x_3 + 3x_4 \\ -y_1 - y_2 - y_3 + 3y_4 \\ -z_1 - z_2 - z_3 + 3z_4 \\ -w_1 - w_2 - w_3 + 3w_4 \end{pmatrix}}_{\sum_{j=1}^{4} c_{4,j} x^j} + \underbrace{\begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{pmatrix}}_{u} \tag{20}$$

where the parameters are selected as $a = 4$, $b = 12$, $c = 5.5$, $m = 1$, and $f_i, d_i (i = 1, 2, 3, 4)$ are unknown. Let the error system as

$$e = Ax^1 + Bx^2 + Cx^3 - Dx^4$$
$$= \begin{pmatrix} \alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3 - m_1 x_4 \\ \alpha_2 y_1 + \beta_2 y_2 + \gamma_2 y_3 - m_2 y_4 \\ \alpha_3 z_1 + \beta_3 z_2 + \gamma_3 z_3 - m_3 z_4 \\ \alpha_4 w_1 + \beta_4 w_2 + \gamma_4 w_3 - m_4 w_4 \end{pmatrix} \tag{21}$$

Similar to the combinatorial inner synchronization of the Yu system, in the simulation process, we take the

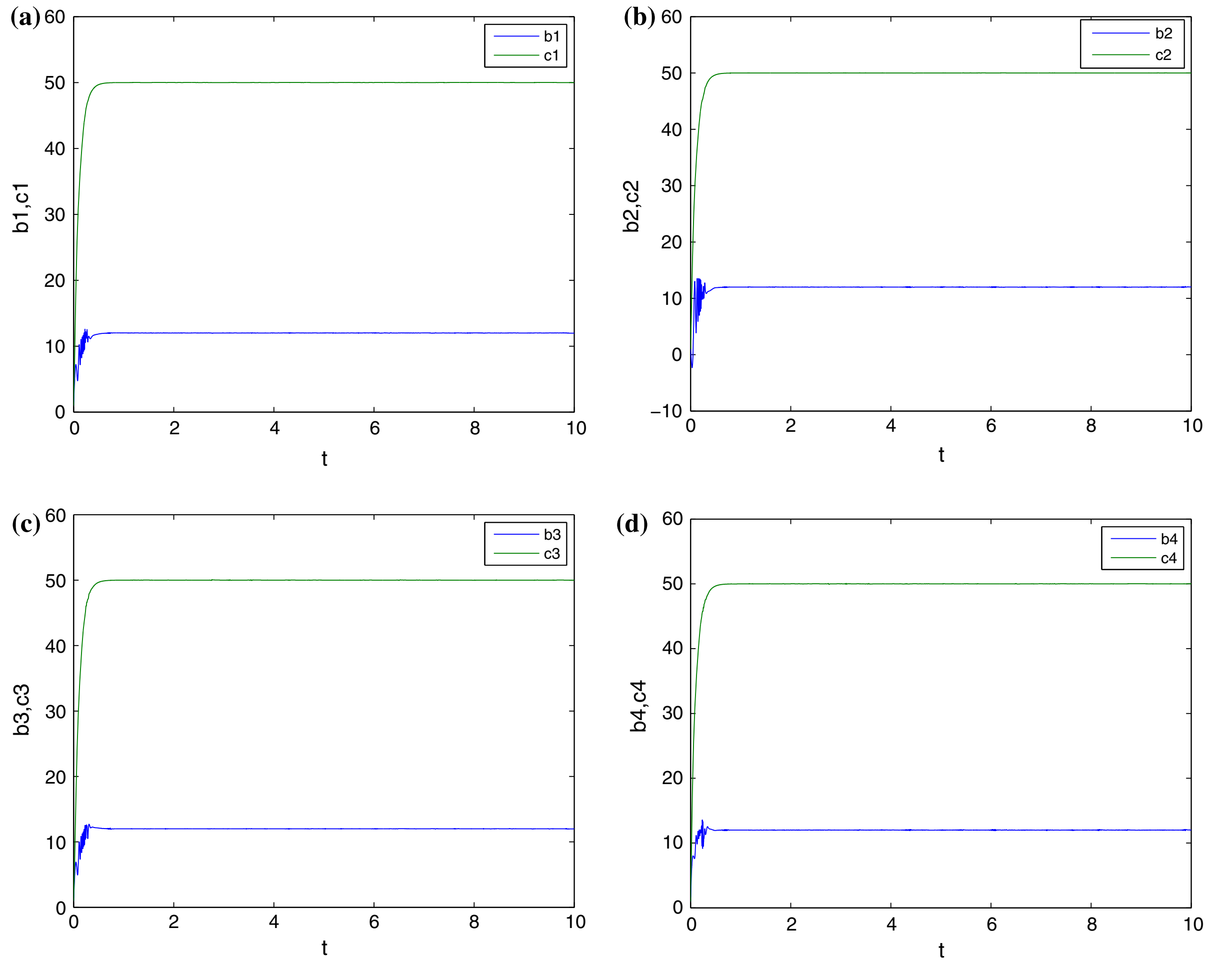

Fig. 6 Time response of the update parameters $b_1, c_1, b_2, c_2, b_3, c_3, b_4, c_4$ for Yu system

matrices as $A = \text{diag}(3, 2, 1, 4)$, $B = \text{diag}(2, 3, 3, 2)$, $C = \text{diag}(3, 2, 4, 3)$, $D = \text{diag}(2, 3, 2, 1)$. The initial conditions for the drive systems and response system can be given as $(x_{10}, y_{10}, z_{10}, w_{10}) = (4, -5, 12, -8)$, $(x_{20}, y_{20}, z_{20}, w_{20}) = (6, 8, -2, 10)$, $(x_{30}, y_{30}, z_{30}, w_{30}) = (5, -4, 8, 16)$ and $(x_{40}, y_{40}, z_{40}, w_{40}) = (2, 6, 10, -4)$. The initial values of estimated parameters are chosen as $(\hat{f}_1, \hat{d}_1) = (1, 1)$, $(\hat{f}_2, \hat{d}_2) = (1, 1)$, $(\hat{f}_3, \hat{d}_3) = (1, 1)$, and $(\hat{f}_4, \hat{d}_4) = (1, 1)$. Meanwhile, we assume $k_1 = 2, k_2 = 3, k_3 = 2, k_4 = 3, \gamma_1 = 1$, $\gamma_2 = 1, \gamma_3 = 0.02, \varepsilon = 0.1$, and the initial values of the feedback gains are selected as $\sigma_{10} = 2, \sigma_{20} = 2, \sigma_{30} = 2, \sigma_{40} = 2$. The time responses of error variables and adaptive parameters $\hat{f}_1, \hat{d}_1, \hat{f}_2, \hat{d}_2, \hat{f}_3, \hat{d}_3, \hat{f}_4, \hat{d}_4$ are shown in Figs. 7, 8 and 9, respectively. From these figures, we can easily get that the synchronization errors e_1, e_2, e_3, e_4 converge to zero very quickly, which means that the combinatorial inner synchronization of Li system with unknown parameters and external disturbances in the same star-like sub-network is realized. As anticipated from theoretic results, the estimated parameters $\hat{f}_i, \hat{d}_i (i = 1, 2, 3, 4)$ finally coincide with the unknown parameters $f_i, d_i (i = 1, 2, 3, 4)$. The reported results remain consistent for other arbitrary initial conditions.

Remark 2 As the fact that each sub-network can perform well alone at the same time, all the unknown parameters of the nodes in multiple sub-networks can be identified exactly in the inner synchronization phase.

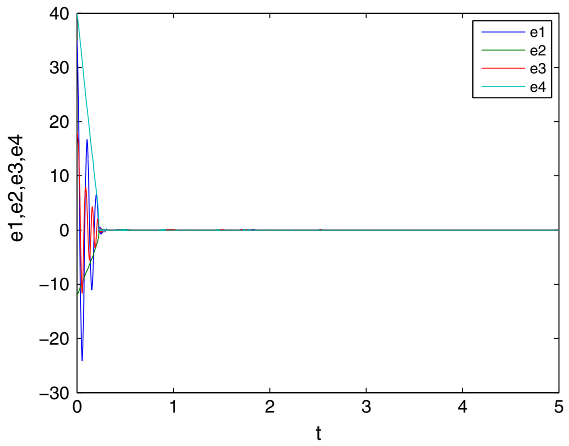

Fig. 7 Time responses of the synchronized error variables for Li system

Unlike the methods in [36–39], the method proposed in this paper is to add an auxiliary item that is related to the corresponding unknown parameters to the parameter update laws, which proves to be not dependent on the selection of initial parameters. That is to say the adaptive method in this paper is free from the restriction that the closer to its real value for the initial parameter, the better of the synchronous performance. From the Figs. 4, 6, 7 and 9, we can see that the method proposed in this paper can perform very well with arbitrary initial values. Furthermore, it needs very short of time in realizing the combinatorial synchronization, and the trajectory is tending to more smooth and steady compared with the previous method as Figs. 10, 11, 12 and 13.

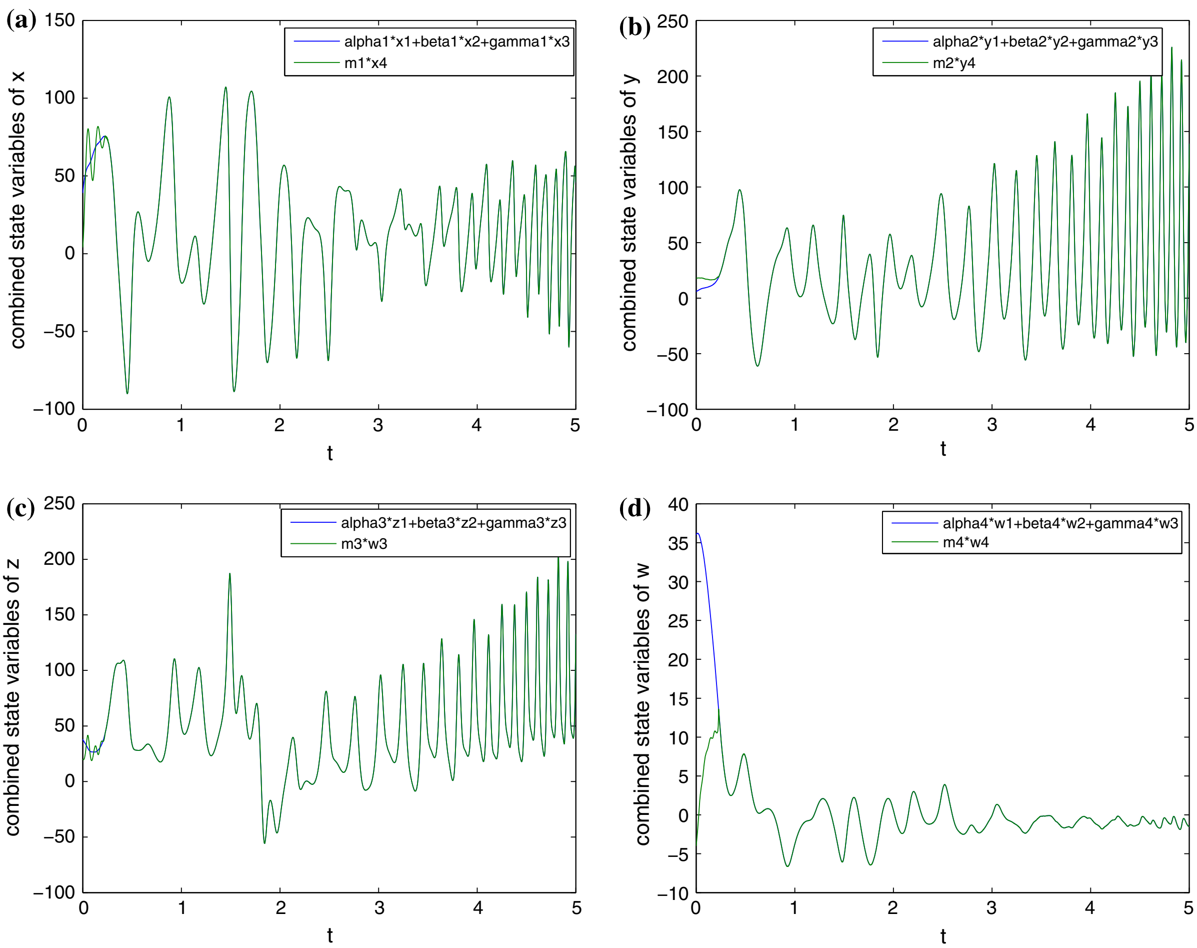

Fig. 8 Time response of the combined variables of Li system

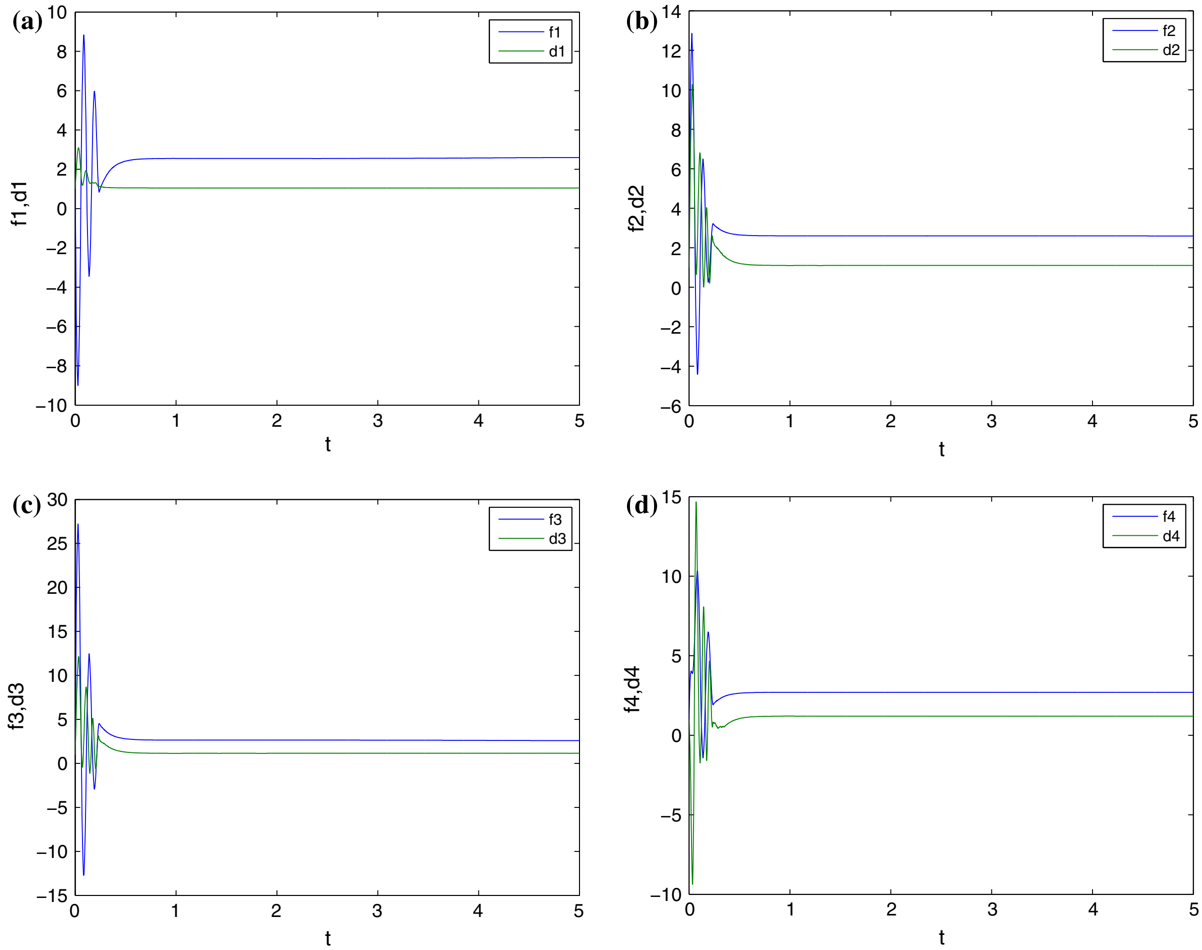

Fig. 9 Time response of the update parameters $f_1, d_1, f_2, d_2, f_3, d_3, f_4, d_4$ for Li system

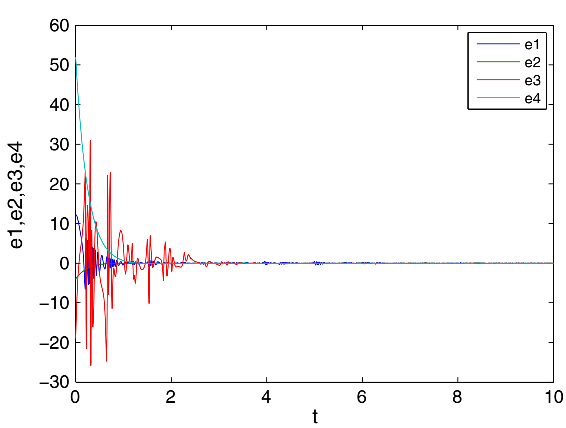

Fig. 10 Time responses of the error variables for Yu system with the general method

4.2 Combinatorial outer synchronization between multiple star-like sub-networks with external disturbances

From the Sect. 4.1, we know that all of the system parameters are identified, but the systems may also be disturbed by some external disturbances unavoidably. For the outer synchronization between different sub-networks, the connection matrix is chosen as

$$C = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{22}$$

For convenience, we just select two different types of systems as the node dynamics in this paper, we assume

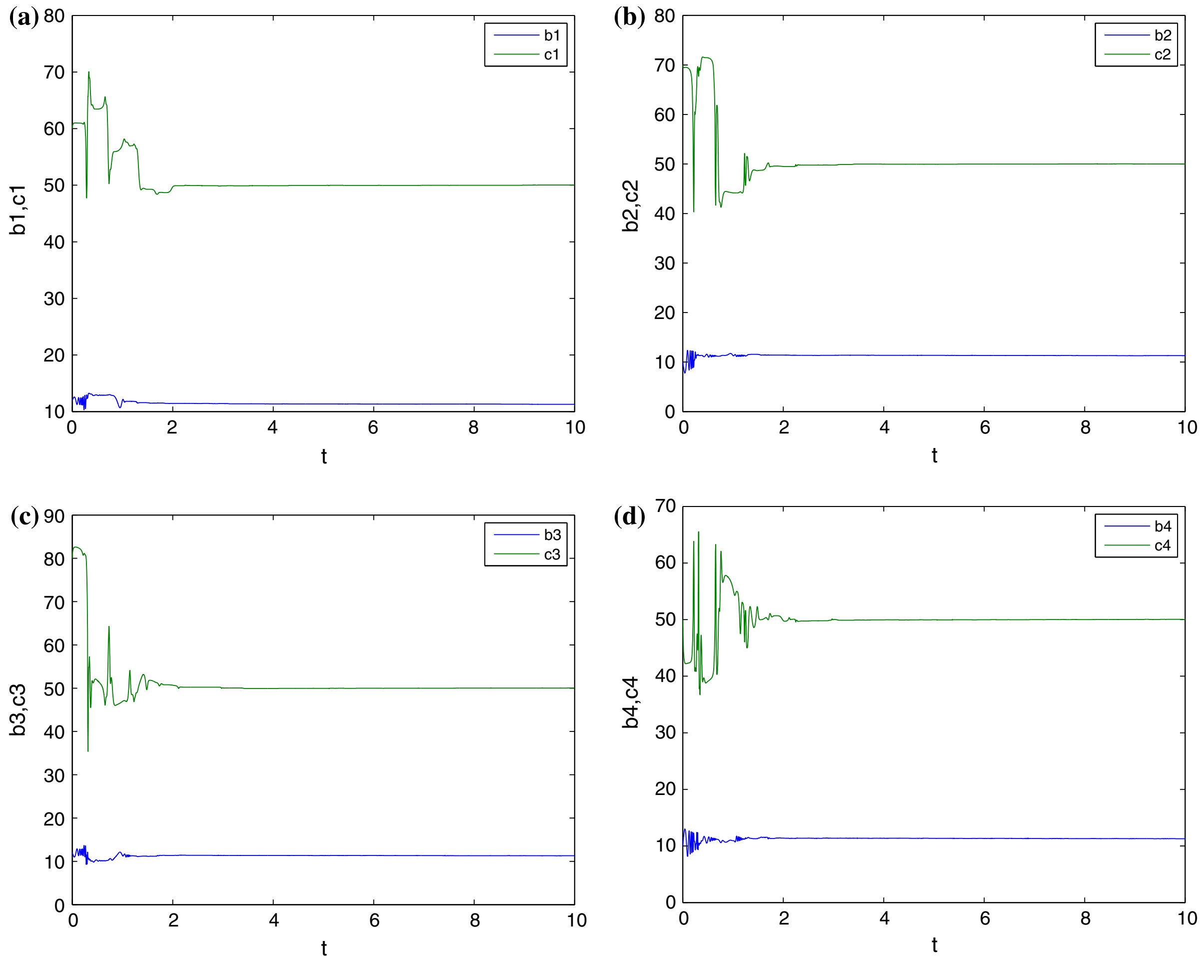

Fig. 11 Time response of the update parameters $b_1, c_1, b_2, c_2, b_3, c_3, b_4, c_4$ for Yu system with the general method

there are three identical sub-networks in which the nodes are constructed by Yu system and only one non-identical sub-network in which the nodes is composed of Li system. The combinatorial outer synchronization of multiple drive systems and one response system can be written as

$$\dot{x}^1 = \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \\ \dot{w}_1 \end{pmatrix} = \underbrace{\begin{pmatrix} -a_1x_1 + y_1z_1 \\ b_1y_1 - x_1z_1 \\ x_1y_1 + d_1w_1 - c_1z_1 \\ x_1z_1 - f_1w_1 \end{pmatrix}}_{f_1(x^1)} + \underbrace{\begin{pmatrix} 0.1\sin(x_1) \\ 0.2\sin(y_1) \\ 0.3\sin(z_1) \\ 0.4\sin(w_1) \end{pmatrix}}_{\Delta f_1(x^1)} + \underbrace{\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \\ w_1 - w_2 \end{pmatrix}}_{\sum \tilde{c}_{1,j}x^j}$$

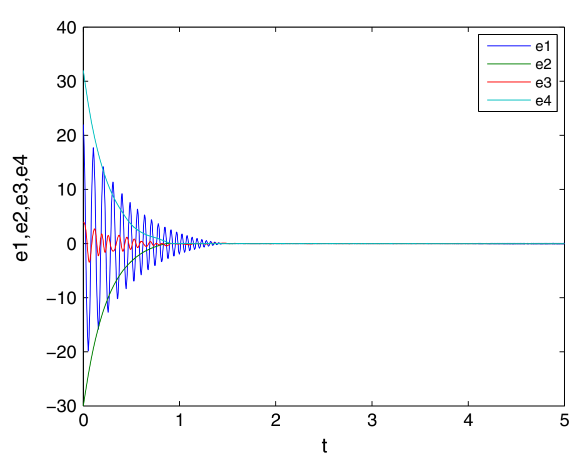

Fig. 12 Time responses of the synchronized error variables for Li system with general method

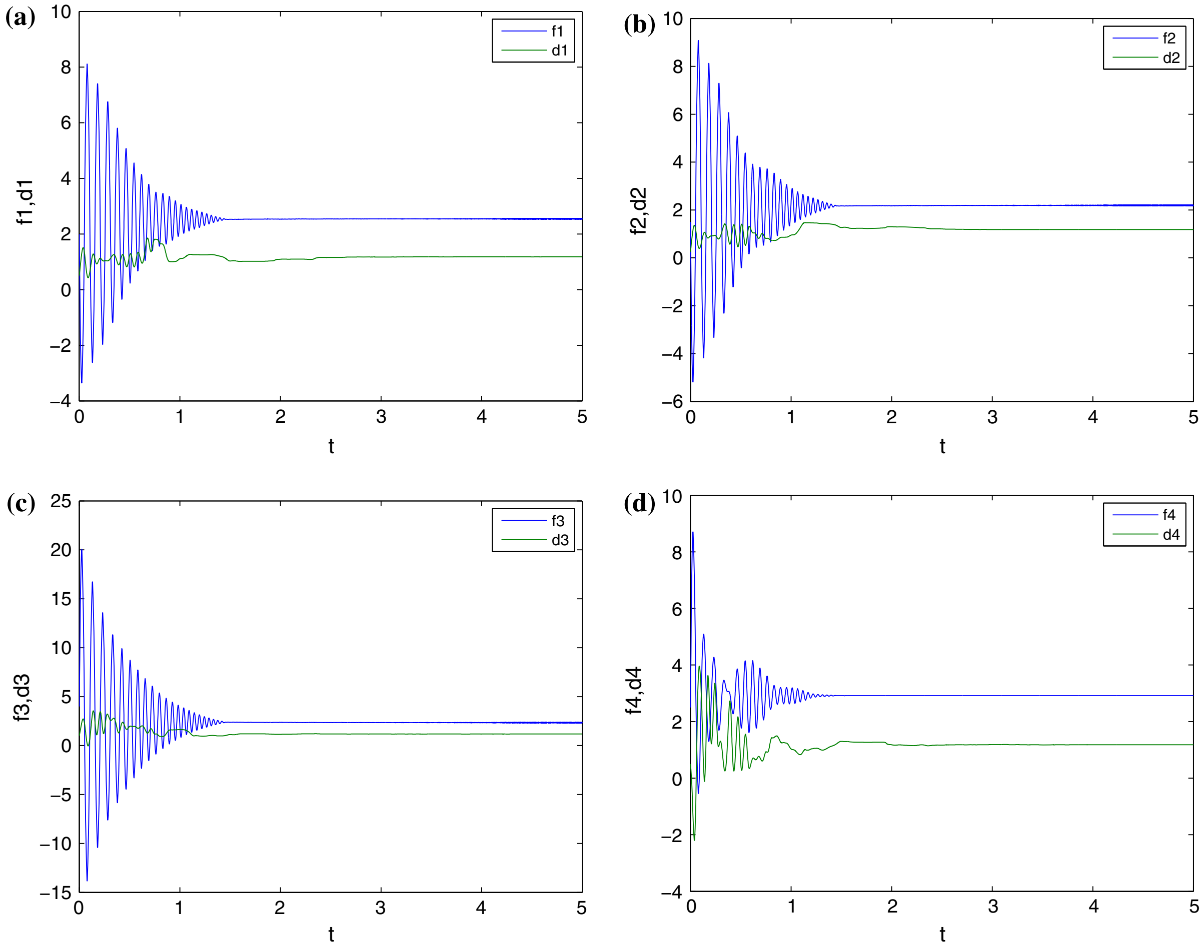

Fig. 13 Time response of the update parameters $f_1, d_1, f_2, d_2, f_3, d_3, f_4, d_4$ for Li system with the general method

$$\dot{x}^2 = \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \\ \dot{z}_2 \\ \dot{w}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -a_1x_2 + y_2z_2 \\ b_1y_2 - x_2z_2 \\ x_2y_2 + d_1w_2 - c_1z_2 \\ x_2z_2 - f_1w_2 \end{pmatrix}}_{f_2(x^2)} + \underbrace{\begin{pmatrix} -0.1\cos(2\pi x_2) \\ -0.2\cos(2\pi y_2) \\ -0.3\cos(2\pi z_2) \\ -0.4\cos(2\pi w_2) \end{pmatrix}}_{\Delta f_2(x^2)} + \underbrace{\begin{pmatrix} x_2 - x_3 \\ y_2 - y_3 \\ z_2 - z_3 \\ w_2 - w_3 \end{pmatrix}}_{\sum \tilde{c}_{2,j}x^j}$$

$$\dot{x}^3 = \begin{pmatrix} \dot{x}_3 \\ \dot{y}_3 \\ \dot{z}_3 \\ \dot{w}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -a_1x_3 + y_3z_3 \\ b_1y_3 - x_3z_3 \\ x_3y_3 + d_1w_3 - c_1z_3 \\ x_3z_3 - f_1w_3 \end{pmatrix}}_{f_3(x^3)} + \underbrace{\begin{pmatrix} 0.1\sin(3\pi x_3) \\ 0.2\cos(3\pi y_3) \\ 0.3\sin(3\pi z_3) \\ 0.4\cos(3\pi w_3) \end{pmatrix}}_{\Delta f_3(x^3)} + \underbrace{\begin{pmatrix} x_3 - x_4 \\ y_3 - y_4 \\ z_3 - z_4 \\ w_3 - w_4 \end{pmatrix}}_{\sum \tilde{c}_{3,j}x^j} \tag{23}$$

$$\dot{x}^4 = \begin{pmatrix} \dot{x}_4 \\ \dot{y}_4 \\ \dot{z}_4 \\ \dot{w}_4 \end{pmatrix} = \underbrace{\begin{pmatrix} a_2x_4 - y_4z_4 + k_2w_4 \\ -b_2y_4 + x_4z_4 \\ -c_2z_4 + x_4y_4 + d_2x_4 \\ -m_2x_4 \end{pmatrix}}_{f_4(x^4)} + \underbrace{\begin{pmatrix} -0.1\cos(4\pi x_4) \\ -0.2\sin(4\pi y_4) \\ 0.3\cos(4\pi z_4) \\ 0.4\sin(4\pi w_4) \end{pmatrix}}_{\Delta f_4(x^4)}$$

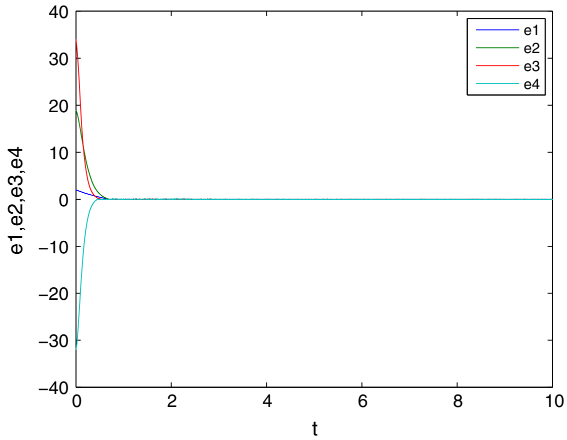

Fig. 14 Time responses of the combinatorial outer synchronization error of multiple star-like sub-networks

$$+\underbrace{\begin{pmatrix} -x_1+x_4 \\ -y_1+y_4 \\ -z_1+z_4 \\ -w_1+w_4 \end{pmatrix}}_{\sum \tilde{c}_{4,j}x^j}+\underbrace{\begin{pmatrix} \tilde{u}_1(t) \\ \tilde{u}_2(t) \\ \tilde{u}_3(t) \\ \tilde{u}_4(t) \end{pmatrix}}_{\tilde{u}} \tag{24}$$

During the simulation process, we take the matrices as $A = \text{diag}(2, 1, 3, 4)$, $B = \text{diag}(2, 3, 1, 2)$, $C = \text{diag}(2, 4, 2, 3)$, $D = \text{diag}(4, 2, 3, 2)$. The choice of initial conditions for the drive systems and response system is arbitrary, which can be taken as $(x_{10}, y_{10}, z_{10}, w_{10}) = (2, 5, 6, -8)$, $(x_{20}, y_{20}, z_{20}, w_{20}) = (5, 10, 4, 9)$, $(x_{30}, y_{30}, z_{30}, w_{30}) = (4, -8, 12, 6)$ and $(x_{40}, y_{40}, z_{40}, w_{40}) = (5, -8, 4, 18)$. The time responses of error variables are shown in Fig. 14. It can be seen that the synchronization errors e_1, e_2, e_3, e_4 converge to zero very quickly, which means that the combinatorial outer synchronization between different multiple star-like sub-networks with external disturbances is realized.

Remark 3 The topological structure of the sub-network in this paper can be extended to other more general network structure, only under the condition of leaving a being synchronized sink node as an interaction center to contact with other ones in different sub-networks.

5 Conclusions

As it is known to all that in complex networks, the nodes in the same or different sub-networks may also need to exchange information with each other in a synchronous way to accomplish a complex task collectively, such as a synchronized multi-robot system, in which each robot with different weighting coefficient interacts with others directly or indirectly, can accomplish a complex multifunctional task in the form of collaboration, which is difficult to be achieved by an individual because the collective behavior of all robots offers more flexibility and maneuverability. Motivated by this discussion, in this paper, we proposed a more general method, which can realize not only the combinatorial inner synchronization within a star-like sub-network, but also the combinatorial outer synchronization between different star-like sub-networks by a suitable switch control scheme. The switch control scheme can be set as a time trigger or an event trigger. Each star-like sub-network has only one response system as an interaction center to contact with not only the nodes within the same sub-network but also other interaction centers in different sub-networks. The sink node is assigned to coordinate with the other nodes, which belong to the same sub-network with the sink node, to realize the combinatorial inner synchronization independently at first. As the fact that every sub-network performs well alone at the same time for the combinatorial inner synchronization, and the time of the synchronization is finite, so we can set a suitable time threshold. Once it achieves at the scheduled time threshold, the interaction center will switch to contact with other interaction centers in different sub-networks for the combinatorial outer synchronization, which means that the proposed switch control scheme is fit for not only the synchronization within a sub-network but also between different sub-networks. By introducing a star-like topological structure that all nodes with identical node dynamics in a sub-network connect with each other in a certain way, only leaving a sole sink node as an interaction center to interchange information with other sink nodes in different sub-network, we only need to one controller for the individual sub-network in realizing the combinatorial inner synchronization and one controller for the combinatorial outer synchronization between different sub-networks. Based on the Lyapunov stability theory, LaSalle's invariance principle, cluster analysis, and pinning control, some sufficient conditions for the combinatorial inner synchronization in a sub-network with unknown parameters and external disturbances, and the combinatorial outer synchronization between multiple sub-networks with external disturbances are derived. Some numerical simulations verify the correctness and effectiveness of our theoretic analysis.

The main features distinguishing our work from the previous ones are that: (1) Most of the previous works focus on only either the inner synchronization within a network or the outer synchronization between different networks, while in reality, there are still some situations that the nodes not only within the same network but also between different sub-networks also need to interchange information with each other in a synchronous way. Therefore, it seems particularly important to study the synchronization of the nodes not only within the same sub-network but also between different sub-networks. (2) As the fact that the nodes in each sub-network have the same system model, and they can synchronize to an identical node, which is taken as a sink node with the inner synchronization, we only have to let the being synchronized sink node as an interaction center to interchange information with other sink nodes in different sub-networks. In fact, we can choose an arbitrary zero-row-sum matrix as a connection matrix for the individual sub-network with only one sink node in connection with other sink nodes. In this paper, we just take the star-like topological structure as an example to testify the feasibility and validity of the proposed scheme. (3) As the previous works need too many controllers, whose numbers are usually the same as the nodes, while in this paper, we only need to one controller for the individual sub-network in realizing the combinatorial inner synchronization, and one controller for the combinatorial outer synchronization between different sub-networks with a suitable switch control scheme. (4) In the parameter estimation stage, unlike the previous methods, the method proposed in this paper is to add an auxiliary item that is related to the corresponding unknown parameters to the parameter update laws, which proves to be not dependent on the selection of initial parameters. That is to say the adaptive method in this paper has good immunity on the selection of initial parameters. (5) The proposed method is simple and typical in the reality and applied to a majority of fields such as technological, social, and biological networks. Furthermore, all of the sub-networks can perform well alone at the same time, and the nodes are setting with different weight coefficients according to its role in the whole synchronous process, which is more in line with the actual and has much more research value in the reality.

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