Cluster Synchronization on Multiple Nonlinearly Coupled Dynamical Subnetworks of Complex Networks With Nonidentical Nodes

Lili Zhou, Chunhua Wang, Sichun Du, and Ling Zhou

***Abstract*— In this paper, cluster synchronization on multiple nonlinearly coupled dynamical subnetworks of complex networks with nonidentical nodes and stochastic perturbations is studied. Based on the general leader–follower's model, an improved network structure model that consists of multiple pairs of matching subnetworks, each of which includes a leaders' subnetwork and a followers' subnetwork, is proposed. Moreover, the dynamical behaviors of the nodes belonging to the same pair of matching subnetworks are identical, while the ones belonging to different pairs of unmatched subnetworks are nonidentical. In this new setting, the aim is to design some suitable adaptive pinning controllers on the chosen nodes of each followers' subnetwork, such that the nodes in each subnetwork can be exponentially synchronized onto their reference state. Then, some cluster synchronization criteria for multiple nonlinearly coupled dynamical subnetworks of complex networks are established, and a pinning control scheme that the nodes with very large or low degrees are good candidates for applying pinning controllers is presented. Suitable adaptive update laws are used to deal with the unknown feedback gains between the pinned nodes and their leaders. Finally, several numerical simulations are given to demonstrate the effectiveness and applicability of the proposed approach.**

***Index Terms*— Exponential convergence, hybrid control, multiple dynamical subnetworks of complex networks, nonidentical nodes, nonlinear coupling.**

I. INTRODUCTION

SINCE the small-world and scale-free network models were constructively proposed in 1998 and 1999, respectively, the study of complex dynamical networks has gained increasing attention. The main reason is that many real systems can be described by complex dynamical networks, such as Internet networks [1], biological networks [2], epidemic spreading networks [3], collaborative networks [4], and social networks [5], to name just a few. Synchronization, as one of an important and interesting collective behavior of complex dynamical networks, has been widely studied in many research and application fields, such as secure communication and information processing. As a result, many different kinds of synchronization patterns have been introduced [6]–[11]. Cluster synchronization means that the complex networks can be split into many clusters, such that the nodes belonging to the same cluster can achieve synchronization individually, but the synchronous states of these clusters are mutually different. In view of its importance in biological science [12], [13] and communication engineering [14], [15], the cluster synchronization of complex networks has been extensively studied.

In the case where the whole network cannot synchronize by its intrinsic structure, some control schemes may be designed to drive the network to synchronization. However, the complex networks usually consist of large numbers of nodes and links, and thus, it is very difficult and unrealistic to control these complex networks by adding the controllers to all nodes. To save control cost, we can divide the larger-scale complex networks into multiple subnetworks, then apply control actions to just a small fraction of network nodes in each subnetwork so as to force the whole network to synchronize. This is what is known as cluster synchronization on multiple subnetworks via pinning control. Nowadays, cluster synchronization of complex networks with pinning control has been widely studied. In [16], the problem of driving a general network to a selected cluster synchronization pattern by means of a pinning control strategy was proposed, and some detailed steps on how to construct the coupling matrix and modify the control strengths were given. The cluster synchronization problem for linearly coupled networks with intermittent pinning controls was investigated in [17]. In [18], the issue of mean square cluster synchronization in directed networks consisting of nonidentical nodes with communication noises was investigated.

However, the pinning schemes proposed in the above literature are all based on the general single-leader–multiple-follower model, in which the leader plays the role of a command generator providing a reference state and it has to be approached by all the followers. Only a fraction of nodes in need of applying controllers in each followers' subnetwork can receive the same information from the sole leader, which restricts the actual application of these pinning schemes to some extent. In addition, it is known to all that the single-leader–multiple-follower network model is very fragile to the deliberate attacks; if the sole leader is attacked, the whole network can be in a mess and all of the nodes cannot reach synchronization anymore. Therefore, a natural question may arise: can we introduce a leaders' network consisting

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of many leaders, such that the followers have much more opportunities to receive information from their leaders, so as to improve the robustness of the complex networks against the deliberate attacks? Recently, a very few works have been devoted to this study. He *et al.* [19] have investigated the finite-time pinning control of second-order multiagent systems with multiple leaders and followers under a fixed directed communication topology, and a new continuous nonlinear pinning control algorithm has been proposed to achieve pinning tracking of the second-order multiagent systems. In [20], the pinning synchronization on complex networks of networks has been investigated. Based on the multiple-leader–multiple-follower model, some synchronization criteria and a suitable pinning scheme have been presented. However, in these two papers, there are still several problems that have not been well resolved, which can be listed as follows.

1) The nodes are all assumed to be identical. In fact, it is not always practical to assume that all network nodes are identical, since some real-world complex networks may consist of different types of nodes [6], [8], [11], [21]–[24].
2) The authors do not consider the influence of stochastic disturbances and time-varying delays. Due to the widespread of the random uncertainties and the finite speeds of transmission, signals traveling through a network are often associated with time-varying delays. In order to make the network model a more realistic representation of real networks, it is very important and necessary to take the influence of these stochastic uncertainties and time-varying delays into account [7]–[9], [24]–[29].
3) Only the nodes belonging to the same subnetwork can connect with each other, while the nodes belonging to different ones cannot connect with the others at all, which does not accord with the real world. As in many practical systems, the nodes belonging to the same or different subnetworks may all need to communicate with each other in a cooperative or even competitive way [17].
4) The proposed pinning schemes are all based on the linearly coupled complex networks, while in many practical cases, it often happens that the coupling scheme is nonlinear [21], [22].
5) The feedback gains of the controllers presented in [20] are much larger than the required values, and this is unrealistic. While the adaptive control approach, known for its main advantage that the control parameters can adjust themselves according to some suitable update laws [30], [31], can deal with this problem very well.

Based on the above consideration, in this paper, we propose an improved multiple-leader–multiple-follower network model to realize the cluster synchronization on multiple nonlinearly coupled dynamical subnetworks of complex networks with stochastic disturbances and time-varying delays. In this improved network model, the complex networks consist of multiple pairs of matching subnetworks, each of which includes a leaders' subnetwork and a followers' subnetwork. The dynamical behaviors of the nodes belonging to the same pair of matching subnetworks are identical, while the ones belonging to different pairs of unmatched subnetworks are nonidentical. Moreover, both the cooperation and competition are all considered simultaneously. The nodes belonging to the same subnetwork can communicate with each other in a cooperative way, while the ones belonging to different subnetworks can communicate with each other in a cooperative or even competitive way. In addition, suitable adaptive control technique is applied to deal with the unknown feedback gains between the pinned nodes and their leaders.

The remainder of this paper is organized as follows. In Section II, an improved network structure model for multiple nonlinearly coupled dynamical subnetworks of complex networks with the consideration of both stochastic disturbances and time-varying delays is presented, and some necessary assumptions, definitions, and lemmas are given. In Section III, some cluster synchronization criteria and a pinning control scheme on multiple nonlinearly coupled dynamical subnetworks of complex networks are proposed. Some numerical simulation examples are provided to validate all of the theoretical results in Section IV. Finally, the conclusion is drawn in Section V.

Notation: In this paper, $\otimes$ represents the Kronecker product. I_N denotes an N-dimensional identity matrix. 1_M and 0_M represent the M-dimensional column vectors with all the elements being 1 or 0, respectively. $\mathbf{0}$ represents a zero matrix. $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ represent the maximum and minimum eigenvalues of the corresponding matrix, respectively. $\|\cdot\|$ stands for the Euclidean vector norm. diag$\{\cdots\}$ represents a diagonal matrix. $E\{\cdot\}$ denotes the mathematical expectation. The superscript T is the transpose. If there are no special instructions, all of the variables are the functions on t. For convenience, the nonlinear function $f(x(t))$ is equal to $f(x)$, time-varying delay $\tau(t)$ is equal to τ_t, and error vector $e(t-\tau_t)$ is equal to e_{τ_t}.

II. Description of the Network Model and Some Preliminaries

In this literature, cluster synchronization on multiple nonlinearly coupled dynamical subnetworks of complex networks with nonidentical nodes and stochastic disturbances will be studied. In our network model, the large-scale complex networks can be divided into two types of networks: a global leaders' network consisting of many leaders and a global followers' network consisting of a host of followers. These leaders and followers, assigned to different pairs of matching subnetworks according to their function of orientation, constitute multiple leaders' subnetworks and follower' subnetworks, respectively. The nodes in each pair of matching subnetworks have identical node dynamics, while the nodes belonging to different pairs of unmatched subnetworks have nonidentical ones. Furthermore, a leader, which represents one side having *a priori* knowledge or the professional skill, or a follower, has to cooperate with the others belonging to the same subnetwork to complete a certain task; while in different subnetworks, it may play the role of a competitor or a partner in completing different tasks. The leaders belonging to the same or different subnetworks can communicate with each other directly, which is the same as the pinned nodes in the global followers' network; but as for the unpinned nodes,

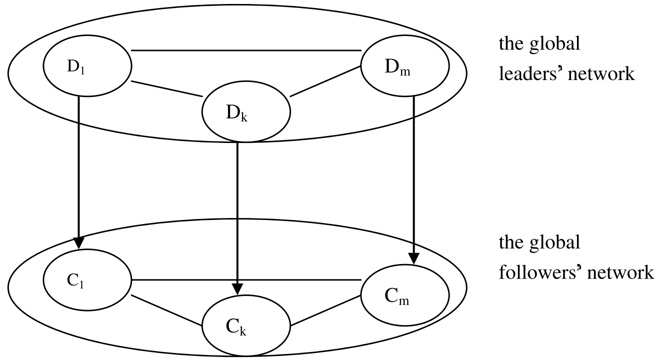

Fig. 1. Network structure diagram constructed by m leaders' subnetworks and m matching followers' subnetworks.

they can only communicate with each other within the same subnetwork. In addition, the leaders are only responsible for providing the related information, such as the necessary priori knowledge or professional skills, to their matching followers; while the responsibilities of the pinned followers are to not only receive information from their leaders but also organize other followers belonging to the same or different subnetworks to complete a certain task in a mutually cooperative way. The corresponding network structure diagram is shown in Fig. 1. Assume that the complex networks are composed of a global followers' network and a global leaders' network, where the global followers' network contains m followers' subnetworks $C_1, C_2, \ldots, C_m$ and the global leaders' network contains m matching leaders' subnetworks $D_1, D_2, \ldots, D_m$. As shown in Fig. 1, the nodes in the kth followers' subnetwork C_k can be represented as $r_{k-1}+1, r_{k-1}+2, \ldots, r_k$ and the ones in the kth matching leaders' subnetwork D_k can be represented as $w_{k-1}+1, w_{k-1}+2, \ldots, w_k$, where $k = 1, 2, \ldots, m$. The kth followers' subnetwork has $N_k = r_k - r_{k-1}$ nodes and the kth matching leaders' subnetwork has $M_k = w_k - w_{k-1}$ nodes, where $r_0 = 0, r_m = N, w_0 = 0$, and $w_m = M$; thus, we have $\sum_{k=1}^{m} N_k = N$ and $\sum_{k=1}^{m} M_k = M$. That is to say, N represents the total number of nodes in the global followers' network and M represents the total number of nodes in the global leaders' network. In order to assign these N followers and M leaders to m followers' subnetworks and m matching leaders' subnetworks, respectively, we can introduce a mapping function, namely, $\mu : \{1, 2, \ldots, N\}$ or $\{1, 2, \ldots, M\} \rightarrow \{1, 2, \ldots, m\}$ to deal with it. If node i belongs to the jth subnetwork, then we have $\mu(i) = \mu_i = j$. Consider the global followers' network consisting of N nonidentical nodes with stochastic disturbances. The dynamic behavior of the ith node can be described by the following stochastic delay differential equation:

$$
\begin{aligned}
dx_i(t) = &\Bigg[A_{\mu_i} x_i(t) + f_{\mu_i}(t, x_i(t), x_i(t-\tau_t)) \\
&+ c \sum_{j=1}^{N} b_{ij}^{(\mu_i)} \Gamma g(x_j(t)) \Bigg] dt \\
&+ \delta(t, x_i(t), x_i(t-\tau_t)) d\omega, \quad i = 1, 2, \ldots, N \qquad (1)
\end{aligned}
$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in R^n$ represents the state vector of node i, A_{μ_i} denotes a negative definite matrix, $f_{\mu_i}(t, x_i(t), x_i(t-\tau_t)) : [0, +\infty] \times R^n \times R^n \rightarrow R^n$ is a continuously differentiable vector function that describes the local dynamics of the nodes in the μ_ith followers' subnetwork, τ_t is a continuously differentiable time-varying delay, c is a coupling strength, $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is an inner coupling matrix that satisfies the condition $\Gamma \geq I_n$. $B^{(\mu_i)} = (b_{ij}^{(\mu_i)}) \in R^{N_{\mu_i} \times N}$ is the coupling configuration matrix representing the topological structure of the μ_ith followers' subnetwork, which can be stated as follows: 1) when $\mu_j = \mu_i$ and $j = i$, then $b_{ij}^{(\mu_i)} = b_{ii}^{(\mu_i)} = -\sum_{k=r_{\mu_i-1}+1, k\neq i}^{r_{\mu_i}} b_{ik}^{(\mu_i)} < 0$; 2) when $\mu_j = \mu_i$ and $j \neq i$, and if the node i receives the information from the node j directly, then $b_{ij}^{(\mu_i)} = b_{ji}^{(\mu_i)} > 0$; otherwise, $b_{ij}^{(\mu_i)} = b_{ji}^{(\mu_i)} = 0$; and 3) when $\mu_j \neq \mu_i$, and if the node i receives the information from the node j directly, then $b_{ij}^{(\mu_i)} \neq 0$ ($b_{ij}^{(\mu_i)} > 0$ or $b_{ij}^{(\mu_i)} < 0$); otherwise, $b_{ij}^{(\mu_i)} = 0$, and it satisfies the condition $\sum_{k=r_{\mu_j-1}+1}^{r_{\mu_j}} b_{ik}^{(\mu_i)} = 0$. $B = [B^{(1)T}\ B^{(2)T}\ \ldots\ B^{(m)T}]^T \in R^{N\times N}$ is the coupling configuration representing the topological structure of the global followers' network, as the global followers' network is undirected, and thus, the matrix B is symmetric. $g(x_j(t)) : R^n \rightarrow R^n$ is a nonlinear coupling function. $\omega(t) = (\omega_1(t), \omega_2(t), \ldots, \omega_n(t))^T$ is an n-dimensional wiener process defined on a complete probability space (Ω, F, P), where Ω is the sample space, F is the σ-algebra of subsets of the sample space, and P is the probability measure on F, and it satisfies the conditions $E\{d\omega\} = 0$ and $E\{(d\omega)^2\} = dt$. $\delta : [0, +\infty] \times R^n \times R^n \rightarrow R^{n\times n}$ is the noise intensity function matrix. This type of stochastic perturbation can be regarded as a result from the occurrence of random uncertainties that affect the dynamic behaviors of the complex networks.

As we know, sometimes network (1) may not reach synchronization by its own, and with the increase of network size, it is not realistic to add controllers to all nodes for realizing the cluster synchronization of network (1). To save control cost, we can apply some control actions to just a small fraction of nodes, which is known as pinning control. Without loss of generality, we can rearrange the order of the nodes in the μ_ith followers' subnetwork, and let the first l_{μ_i} nodes be controlled. Therefore, the pinning-controlled μ_ith followers' subnetwork with the influence of stochastic factors can be written as

$$
\begin{aligned}
dx_i^{(\mu_i)}(t) = &\Bigg[A_{\mu_i} x_i^{(\mu_i)}(t) + f_{\mu_i}\big(t, x_i^{(\mu_i)}(t), x_i^{(\mu_i)}(t-\tau_t)\big) \\
&+ c \sum_{j=1}^{N} b_{ij}^{(\mu_i)} \Gamma g(x_j(t)) + u_i^{(\mu_i)}(t) \Bigg] dt \\
&+ \delta\big(t, x_i^{(\mu_i)}(t), x_i^{(\mu_i)}(t-\tau_t)\big) d\omega \\
&\qquad i = r_{\mu_i-1}+1, \ldots, l_{\mu_i} \\
dx_i^{(\mu_i)}(t) = &\Bigg[A_{\mu_i} x_i^{(\mu_i)}(t) + f_{\mu_i}\big(t, x_i^{(\mu_i)}(t), x_i^{(\mu_i)}(t-\tau_t)\big) \\
&+ c \sum_{j=1}^{N} b_{ij}^{(\mu_i)} \Gamma g(x_j(t)) \Bigg] dt \\
&+ \delta\big(t, x_i^{(\mu_i)}(t), x_i^{(\mu_i)}(t-\tau_t)\big) d\omega \\
&\qquad i = l_{\mu_i}+1, \ldots, r_{\mu_i} \qquad (2)
\end{aligned}
$$

where $u_i^{(\mu_i)}(t)$ is a designed controller for the pinned nodes in the μ_ith followers' subnetwork.

Consider the global leaders' network consisting of M leaders, where the dynamics of the leaders are identical if they belong to the same leaders' subnetwork and nonidentical if they belong to different leaders' subnetworks. Each local leaders' subnetwork has $M_k (k = 1, 2, \ldots, m)$ leaders, and then, the dynamics of the leaders can be described as

$$ds_i^{(k)}(t) = \Bigg[A_k s_i^{(k)}(t) + f_k\big(t, s_i^{(k)}(t), s_i^{(k)}(t-\tau_t)\big) + c\sum_{j=1}^{M} h_{ij}^{(k)}\Gamma g(s_j(t))\Bigg]dt + \delta\big(t, s_i^{(k)}(t), s_i^{(k)}(t-\tau_t)\big)d\omega \qquad i = \varpi_{k-1}+1, \ldots, \varpi_k \tag{3}$$

where $s_i^{(k)}(t) = (s_{i1}^{(k)}(t), s_{i2}^{(k)}(t), \ldots, s_{in}^{(k)}(t))^T \in R^n$ is the state vector of the ith leader. $H^{(k)} = (h_{ij}^{(k)}) \in R^{M_k \times M}$ is the coupling configuration matrix representing the topological structure of the kth leaders' subnetwork, and the definitions of the matrices $H^{(k)}$ and H are the same as that of the matrices $B^{(k)}$ and B. The same intensity function $\delta(\cdot,\cdot,\cdot)$ means that all nodes considered are in the same environment.

In order to realize the cluster synchronization of complex networks (2) and (3), some assumptions must be noted as follows.

Assumption A1: There exist a constant matrix K and a positive-definite matrix $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, such that f satisfies the following inequality:

$$(x-y)^T(f(x)-f(y)) \le (x-y)^T K\Gamma(x-y) \quad \forall x, y \in R^n.$$

Assumption A2: For the vector-valued function $f(x, \bar{x})$, there exist two positive constants α_1 and β_1, such that

$$\begin{aligned}(x_1-y_1)^T&[f(x_1,\bar{x}_1)-f(y_1,\bar{y}_1)]\\ &\le \alpha_1(x_1-y_1)^T(x_1-y_1)\\ &\quad +\beta_1(\bar{x}_1-\bar{y}_1)^T(\bar{x}_1-\bar{y}_1) \quad \forall x_1, y_1, \bar{x}_1, \bar{y}_1 \in R^n.\end{aligned}$$

Assumption A3: There exist two nonnegative constants p_1 and q_1, and δ is locally Lipschitz continuous, such that

$$\begin{aligned}&\mathrm{trace}\{[\delta(t,x_1,y_1)-\delta(t,x_2,y_2)]^T[\delta(t,x_1,y_1)-\delta(t,x_2,y_2)]\}\\ &\le p_1(x_1-x_2)^T(x_1-x_2)+q_1(y_1-y_2)^T(y_1-y_2)\\ &\qquad \forall x_1, x_2, y_1, y_2 \in R^n, \quad t \in [0, +\infty].\end{aligned}$$

Assumption A4 [32]: A nonlinear function $g(\cdot): R \to R$ is said to belong to the acceptable nonlinear coupling function class, denoted by $g(\cdot) \in \mathrm{NCF}(\vartheta, \varepsilon)$, if there exist two nonnegative scalars ϑ and ε, such that $g(x) - \vartheta x$ satisfies the Lipschitz condition $|g(x_1) - g(x_2) - \vartheta(x_1 - x_2)| \le \varepsilon|x_1 - x_2|$ for all $x_1, x_2 \in R$.

Remark 1: Since the function $g(\cdot) \in \mathrm{NCF}(\vartheta, \varepsilon)$ is the restriction of the oscillatory amplitude of $g(x)$ around the linear function ϑx, a nonlinear function can be made to approach a linear function by taking a large ϑ and small ε. The nonlinear function $g(x)$ can, therefore, be decomposed into the linear part ϑx and the oscillatory part $r(x) = g(x) - \vartheta x$. Obviously, $g(x)$ satisfies $\vartheta - \varepsilon \le (g(x_1) - g(x_2))/(x_1 - x_2) \le \vartheta + \varepsilon$ for all $x_1, x_2 \in R$.

Assumption A5: $\tau(t)$ is a bounded and continuously differentiable function, satisfying $0 < \tau(t) < \tau_0$ and $0 \le \dot{\tau}(t) \le \varepsilon < 1$. Clearly, this assumption is justified when $\tau(t)$ is a constant.

Remark 2: The constraint for the function f in Assumption A1 or A2 is very gentle and it is much weaker than the Lipschitz condition.

In order to derive our main results, the following basic definition and some useful lemmas are needed.

Definition 1: A complex network with N nodes is said to realize cluster synchronization with the exponential rate of convergence, if the node-set $\{1, 2, \ldots, N\}$ is split into m nonempty subsets $C_1, C_2, \ldots, C_m$, and then for arbitrary nodes i and j, if and only if there are some constants $M_k > 0$ and $\mu > 0$, such that for any initial conditions, inequalities $E\{\|x_i^{(k)}(t) - x_j^{(k)}(t)\|^2\} \le M_k \exp(-\mu t)$ hold for $t \ge 0$, where $i, j \in \{1, 2, \ldots, N\}$ and $i \ne j, k \in \{1, 2, \ldots, m\}$, and then, we say that these error states converge to 0 at an exponential rate.

Remark 3: The general complete synchronization is having all the nodes synchronized to an isolated node in the complex networks, while the cluster synchronization, as a special form of the complete synchronization, means that the nodes can synchronize to an isolated node in each subnetwork, but there is no synchronization among the nodes belonging to different subnetworks.

Lemma 1 (Schur Complement [33]): The following linear matrix inequality:

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{12}^T & s_{22} \end{pmatrix} < 0$$

where $s_{11} = s_{11}^T$ and $s_{22} = s_{22}^T$ are equivalent to one of the following conditions.

1) $s_{11} < 0, \ s_{22} - s_{12}^T s_{11}^{-1} s_{12} < 0$.
2) $s_{22} < 0, \ s_{11} - s_{12} s_{22}^{-1} s_{12}^T < 0$.

Lemma 2: Let $1_n = (1, 1, \ldots, 1)^T, I_n = \mathrm{diag}\{1, 1, \ldots, 1\} \in R^n$, and $Q = (q_{ij}) = I_n - (1/N)1_n \cdot 1_n^T$. For all zero-row-sum matrices $M \in R^{m \times n}$ and $\theta > 0$, we have

$$x^T M y = x^T M Q y \le \frac{1}{2}\left(\frac{1}{\theta}x^T M M^T x + \theta y^T Q y\right).$$

Lemma 3 [34]: If $Q \in R^{n \times n}$ is a symmetric matrix and satisfies the condition $q_{ii} = -\sum_{j=1, j \ne i}^{n} q_{ij}, i, j = 1, 2, \ldots, n$, then $u^T Q v = \sum_{i=1}^{n}\sum_{j=1}^{n} u_i q_{ij} v_j = -\sum_{j>i} q_{ij}(u_i - u_j)(v_i - v_j)$, for all vectors $u = (u_1, u_2, \ldots, u_n)^T$ and $v = (v_1, v_2, \ldots, v_n)^T$.

III. Main Results for Cluster Synchronization on Multiple Nonlinearly Coupled Dynamical Subnetworks of Complex Networks

In this section, we will establish some cluster synchronization criteria and propose a pinning control scheme on how to select pinned nodes with lower cost to reach the cluster synchronization on multiple nonlinearly coupled dynamical subnetworks of complex networks.

A. Cluster Synchronization Criteria on Multiple Nonlinearly Coupled Dynamical Subnetworks of Complex Networks With Nonidentical Nodes

In this improved multiple-leader–multiple-follower network model, there are many leaders in each leaders' subnetwork, from which the pinned nodes in the matching followers' subnetwork can receive the information, and the average state of these leaders is regarded as the reference state. In this paper, we aim to analytically prove that all the nodes in each pair of matching subnetworks can be synchronized to their reference state. Let $\bar{s}_k(t) = (1/M_k)\sum_{j=\omega_{k-1}+1}^{\omega_k} s_j^{(k)}(t)$ be the average state of all the leaders in the kth leaders' subnetwork. As the fact that $H^{(k)} = (h_{ij}^{(k)}) \in R^{M_k\times M}$ is a zero-row-sum and zero-column-sum matrix, then we have $\sum_{j=\omega_{k-1}+1}^{\omega_k}\sum_{i=1}^{M} h_{ji}^{(k)}\Gamma g(s_i(t)) = \sum_{i=1}^{M}\sum_{j=\omega_{k-1}+1}^{\omega_k} h_{ji}^{(k)}\Gamma g(s_i(t)) = 0$. Therefore, the dynamics of $\bar{s}_k(t)$ can be described by

$$d\bar{s}_k(t) = A_k\bar{s}_k(t)dt + \frac{1}{M_k}\sum_{j=\omega_{k-1}+1}^{\omega_k} f_k\big(t, s_j^{(k)}(t), s_{j\tau_t}^{(k)}\big)dt + \frac{1}{M_k}\sum_{j=\omega_{k-1}+1}^{\omega_k}\delta\big(t, s_j^{(k)}(t), s_{j\tau_t}^{(k)}\big)d\omega \quad k = 1, 2, \ldots, m. \tag{4}$$

Let $e_i^{(sk)}(t) = s_i^{(k)}(t) - \bar{s}_k(t)$ be the error state between the ith leader and the average state of the leaders in the kth leaders' subnetwork, where $i = \omega_{k-1}+1, \ldots, \omega_k$; $k = 1, 2, \ldots, m$. Subtracting (3) from (4) yields the following error dynamical network:

$$\begin{aligned} de_i^{(sk)}(t) &= \left[A_k e_i^{(sk)}(t) + f_k\big(t, s_i^{(k)}(t), s_{i\tau_t}^{(k)}\big) + c\sum_{j=1}^{M} h_{ij}^{(k)}\Gamma g(s_j(t)) - \frac{1}{M_k}\sum_{j=\omega_{k-1}+1}^{\omega_k} f_k\big(t, s_j^{(k)}(t), s_{j\tau_t}^{(k)}\big)\right]dt \\ &\quad + \left[\delta\big(t, s_i^{(k)}(t), s_{i\tau_t}^{(k)}\big) - \frac{1}{M_k}\sum_{j=\omega_{k-1}+1}^{\omega_k}\delta\big(t, s_j^{(k)}(t), s_{j\tau_t}^{(k)}\big)\right]d\omega \\ &\qquad i = \omega_{k-1}+1, \omega_{k-1}+2, \ldots, \omega_k. \end{aligned} \tag{5}$$

Next, two theorems are established to derive the cluster synchronization criteria for the nonlinearly coupled leaders' subnetwork (3) and followers' subnetwork (2), respectively.

Theorem 1: Assume that Assumptions A1–A5 hold, for given scalars μ, θ, ς, and k_1, the leaders in the global leaders' network are all exponentially synchronized to their corresponding reference states if there exists a positive-definite and symmetric matrix $R_1 \in R^{n\times n}$, such that the following conditions hold:

$$\begin{aligned} \Sigma_1 &= c\left[\vartheta\lambda_2(H) + \frac{1}{2\theta}\lambda_{\max}(HH^T) + \varepsilon^2\theta\left(1 - \frac{1}{M}\right)\right]I_n \\ &\quad + \left(\alpha + 2p + \frac{\mu}{2}\right)I_n + k_1\exp(\mu\tau_0)R_1 < 0 \\ \Sigma_3 &= (\beta + 2q)I_n - k_1(1-\varsigma)R_1 < 0 \end{aligned} \tag{6}$$

where $\{\alpha, \beta\} = \max\{\alpha_i, \beta_i | i = 1, 2, \ldots, M\}$, $\{p, q\} = \max\{p_i, q_i | i = 1, 2, \ldots, M\}$, and $\lambda_2(H)$ represents the second largest eigenvalue of the matrix H and satisfies the condition that $\lambda_2(H) = \max_{x^T 1_M = 0, x\neq 0_M} x^T Hx/x^T x$ [35].

Proof: Consider the Lyapunov functional candidate for the error system (5)

$$\begin{aligned} V_1(t, e^{(s)}(t)) &= \frac{1}{2}\sum_{k=1}^{m}\sum_{i=\omega_{k-1}+1}^{\omega_k} e_i^{(sk)T}(t)e_i^{(sk)}(t)\exp(\mu t) \\ &\quad + k_1\sum_{k=1}^{m}\sum_{i=\omega_{k-1}+1}^{\omega_k}\int_{t-\tau_t}^{t} e_i^{(sk)T}(\upsilon)R_1 e_i^{(sk)}(\upsilon) \\ &\quad \times\exp(\mu(\upsilon + \tau_t))d\upsilon \end{aligned} \tag{7}$$

where $k_1 > 0$. Taking the stochastic differential $dV_1(t, e^{(s)}(t))$ along the trajectories of (5), which can be given as

$$dV_1(t, e^{(s)}(t)) = LV_1(t, e^{(s)}(t))dt + \sum_{k=1}^{m}\sum_{i=\omega_{k-1}+1}^{\omega_k} e_i^{(sk)T}(t)X_i^{(k)}\exp(\mu t)d\omega \tag{8}$$

where $X_i^{(k)} = \delta(t, s_i^{(k)}, s_{i\tau_t}^{(k)}) - (1/M_k)\sum_{j=\omega_{k-1}+1}^{\omega_k}\delta(t, s_j^{(k)}, s_{j\tau_t}^{(k)})$. For convenience, let $LV_1(t, e^{(s)}(t)) = LV_1'(t, e^{(s)}(t))\exp(\mu t)$, and the weak infinitesimal operator L [36] is given by

$$\begin{aligned} &LV_1'(t, e^{(s)}(t)) \\ &= \sum_{k=1}^{m}\sum_{i=\omega_{k-1}+1}^{\omega_k} e_i^{(sk)T}(t)A_k e_i^{(sk)}(t) \\ &\quad + \sum_{k=1}^{m}\sum_{i=\omega_{k-1}+1}^{\omega_k} e_i^{(sk)T}(t)\big[f_k\big(t, s_i^{(k)}, s_{i\tau_t}^{(k)}\big) - f_k(t, \bar{s}_k, \bar{s}_{k\tau_t})\big] \\ &\quad + \sum_{k=1}^{m}\sum_{i=\omega_{k-1}+1}^{\omega_k} ce_i^{(sk)T}(t)\sum_{j=1}^{M} h_{ij}^{(k)}\Gamma g(s_j) \\ &\quad - \sum_{k=1}^{m}\frac{1}{M_k}\sum_{i,j=\omega_{k-1}+1}^{\omega_k} e_i^{(sk)T}(t) \\ &\quad \times\big[f_k\big(t, s_j^{(k)}, s_{j\tau_t}^{(k)}\big) - f_k(t, \bar{s}_k, \bar{s}_{k\tau_t})\big] \\ &\quad + \frac{1}{2}\sum_{k=1}^{m}\sum_{i=\omega_{k-1}+1}^{\omega_k}\mathrm{trace}\big(X_i^{(k)T}X_i^{(k)}\big) \\ &\quad + \frac{\mu}{2}\sum_{k=1}^{m}\sum_{i=\omega_{k-1}+1}^{\omega_k} e_i^{(sk)T}(t)e_i^{(sk)}(t) \\ &\quad + k_1\sum_{k=1}^{m}\sum_{i=\omega_{k-1}+1}^{\omega_k} e_i^{(sk)T}(t)R_1 e_i^{(sk)}(t)\exp(\mu\tau_t) \\ &\quad - k_1\sum_{k=1}^{m}\sum_{i=\omega_{k-1}+1}^{\omega_k}(1-\dot{\tau}_t)e_{i\tau_t}^{(sk)T}R_1 e_{i\tau_t}^{(sk)} \\ &\leq \sum_{k=1}^{m}\sum_{i=\omega_{k-1}+1}^{\omega_k} e_i^{(sk)T}(t)A_k e_i^{(sk)}(t) + V_2 + V_3 + V_4 \\ &\quad + \frac{\mu}{2}\sum_{k=1}^{m}\sum_{i=\omega_{k-1}+1}^{\omega_k} e_i^{(sk)T}(t)e_i^{(sk)}(t) \\ &\quad + k_1\sum_{i=1}^{M}\big[e_i^{(s)T}(t)R_1 e_i^{(s)}(t)\exp(\mu\tau_t) - (1-\dot{\tau}_t)e_{i\tau_t}^{(s)T}R_1 e_{i\tau_t}^{(s)}\big]. \end{aligned} \tag{9}$$

According to Assumption A2, we can obtain

$$V_2 = \sum_{k=1}^{m} \sum_{i=w_{k-1}+1}^{w_k} e_i^{(sk)T}(t)\big[f_k\big(t, s_i^{(k)}, s_{i\tau_t}^{(k)}\big) - f_k\big(t, \bar{s}_k, \bar{s}_{k\tau_t}\big)\big]$$
$$\leq \sum_{i=1}^{M} \alpha_i e_i^{(s)T}(t) e_i^{(s)}(t) + \sum_{i=1}^{M} \beta_i e_{i\tau_t}^{(s)T} e_{i\tau_t}^{(s)}. \quad (10)$$

Let

$$e^{(s)k}(t) = \begin{pmatrix} e_{1k}^{(s)}(t) \\ e_{2k}^{(s)}(t) \\ \vdots \\ e_{Mk}^{(s)}(t) \end{pmatrix}, \quad \bar{x}^k(t) = \begin{pmatrix} x_{\mu_1 k}(t) \\ x_{\mu_2 k}(t) \\ \vdots \\ x_{\mu_M k}(t) \end{pmatrix}$$
$$\tilde{g}(x^k(t)) = \begin{pmatrix} g(x_{1k}(t)) \\ g(x_{2k}(t)) \\ \vdots \\ g(x_{Mk}(t)) \end{pmatrix}, \quad \tilde{g}(\bar{x}^k(t)) = \begin{pmatrix} g(x_{\mu_1 k}(t)) \\ g(x_{\mu_2 k}(t)) \\ \vdots \\ g(x_{\mu_M k}(t)) \end{pmatrix}$$
$$\tilde{r}(x^k(t)) = \begin{pmatrix} r(x_{1k}(t)) \\ r(x_{2k}(t)) \\ \vdots \\ r(x_{Mk}(t)) \end{pmatrix}, \quad \tilde{r}(\bar{s}^k(t)) = \begin{pmatrix} r(s_{\mu_1 k}(t)) \\ r(s_{\mu_2 k}(t)) \\ \vdots \\ r(s_{\mu_M k}(t)) \end{pmatrix}.$$

Note that $g(\cdot) \in \mathrm{NCF}(\vartheta, \varepsilon)$, we have

$$V_3 = \sum_{k=1}^{m} \sum_{i=w_{k-1}+1}^{w_k} c e_i^{(sk)T}(t) \sum_{j=1}^{M} h_{ij}^{(k)} \Gamma g(s_j)$$
$$= c \sum_{k=1}^{M} e_k^{(s)T}(t) \sum_{j=1}^{M} h_{kj}^{(\mu_k)} \Gamma[g(s_j) - g(\bar{s}_{\mu_j})]$$
$$= c \sum_{k=1}^{n} \gamma_k e^{(s)kT}(t) H[\tilde{g}(s^k) - \tilde{g}(\bar{s}^k)]$$
$$= c\vartheta \sum_{k=1}^{n} \gamma_k e^{(s)kT}(t) H e^{(s)k}(t)$$
$$+ c \sum_{k=1}^{n} \gamma_k e^{(s)kT}(t) H[\tilde{r}(s^k) - \tilde{r}(\bar{s}^k)]. \quad (11)$$

By using Lemmas 2 and 3, one can obtain

$$c \sum_{k=1}^{n} \gamma_k e^{(s)kT}(t) H[\tilde{r}(s^k) - \tilde{r}(\bar{s}^k)]$$
$$\leq \frac{c}{2} \sum_{k=1}^{n} \frac{\gamma_k}{\theta} e^{(s)kT}(t) H H^T e^{(s)k}(t)$$
$$+ \frac{c}{2} \sum_{k=1}^{n} \gamma_k \theta [\tilde{r}(s^k) - \tilde{r}(\bar{s}^k)]^T Q [\tilde{r}(s^k) - \tilde{r}(\bar{s}^k)]$$
$$\leq \frac{c}{2} \sum_{k=1}^{n} \frac{\gamma_k}{\theta} e^{(s)kT}(t) H H^T e^{(s)k}(t)$$
$$- c \sum_{k=1}^{n} \gamma_k \theta \sum_{j>i} q_{ij} \{[\tilde{r}(s_{ik}) - \tilde{r}(\bar{s}_{\mu_i k})]^2 + [\tilde{r}(s_{jk}) - \tilde{r}(\bar{s}_{\mu_j k})]^2\}$$
$$\leq \frac{c}{2} \sum_{k=1}^{n} \frac{\gamma_k}{\theta} e^{(s)kT}(t) H H^T e^{(s)k}(t)$$
$$- c\varepsilon^2 \sum_{k=1}^{n} \gamma_k \theta \sum_{j>i} q_{ij} \big[e_{ik}^{(s)}(t)^2 + e_{jk}^{(s)}(t)^2\big]$$
$$= \frac{c}{2} \sum_{k=1}^{n} \frac{\gamma_k}{\theta} e^{(s)kT}(t) H H^T e^{(s)k}(t)$$
$$+ c\varepsilon^2 \sum_{k=1}^{n} \gamma_k \theta \left(1 - \frac{1}{M}\right) e^{(s)kT}(t) e^{(s)k}(t)$$
$$= c \sum_{k=1}^{n} \gamma_k e^{(s)kT}(t) \left[\frac{1}{2\theta} H H^T + \varepsilon^2 \theta \left(1 - \frac{1}{M}\right) I_M\right] e^{(s)k}(t). \quad (12)$$

Therefore, (11) can be written as

$$V_3 \leq c \sum_{k=1}^{n} \gamma_k e^{(s)kT}(t) \Big[\vartheta H + \frac{1}{2\theta} H H^T + \varepsilon^2 \theta \left(1 - \frac{1}{M}\right) I_M\Big] e^{(s)k}(t). \quad (13)$$

In view of Assumption A3, we can easily get

$$V_4 = \frac{1}{2} \sum_{k=1}^{m} \sum_{i=w_{k-1}+1}^{w_k} \mathrm{trace}\big(X_i^{(k)T} X_i^{(k)}\big)$$
$$\leq \sum_{k=1}^{m} \frac{1}{2M_k} \sum_{i,j=w_{k-1}+1}^{w_k} p_i \big(s_i^{(k)} - s_j^{(k)}\big)^T \big(s_i^{(k)} - s_j^{(k)}\big)$$
$$+ \sum_{k=1}^{m} \frac{1}{2M_k} \sum_{i,j=w_{k-1}+1}^{w_k} q_i \big(s_{i\tau_t}^{(k)} - s_{j\tau_t}^{(k)}\big)^T \big(s_{i\tau_t}^{(k)} - s_{j\tau_t}^{(k)}\big)$$
$$\leq \sum_{k=1}^{m} \frac{1}{M_k} \sum_{i,j=w_{k-1}+1}^{w_k} p\big[e_i^{(sk)T}(t) e_i^{(sk)}(t) + e_j^{(sk)T}(t) e_j^{(sk)}(t)\big]$$
$$+ \sum_{k=1}^{m} \frac{1}{M_k} \sum_{i,j=w_{k-1}+1}^{w_k} q\big(e_{i\tau_t}^{(sk)T} e_{i\tau_t}^{(sk)} + e_{j\tau_t}^{(sk)T} e_{j\tau_t}^{(sk)}\big)$$
$$= \sum_{i=1}^{M} \big[2p e_i^{(s)T}(t) e_i^{(s)}(t) + 2q e_{i\tau_t}^{(s)T} e_{i\tau_t}^{(s)}\big] \quad (14)$$

where $\{p, q\} = \max\{p_i, q_i | i = 1, 2, \ldots, M\}$.

Substituting inequalities (10), (13), and (14) into (9), and considering Assumption A5, one can easily obtain that

$$LV_1'(t, e^{(s)}(t))$$
$$\leq \sum_{i=1}^{M} e_i^{(s)T}(t) A_{\mu_i} e_i^{(s)}(t) + \sum_{i=1}^{M} \beta_i e_{i\tau_t}^{(s)T} e_{i\tau_t}^{(s)}$$
$$+ \sum_{i=1}^{M} \alpha_i e_i^{(s)T}(t) e_i^{(s)}(t) + \frac{\mu}{2} \sum_{k=1}^{n} e^{(s)kT}(t) e^{(s)k}(t)$$
$$+ c \sum_{k=1}^{n} \gamma_k e^{(s)kT}(t) \left[\vartheta H + \frac{1}{2\theta} H H^T + \varepsilon^2 \theta \left(1 - \frac{1}{M}\right) I_M\right]$$
$$\times e^{(s)k}(t)$$
$$+ \sum_{k=1}^{n} \big[2p e^{(s)kT}(t) e^{(s)k}(t) + 2q e_{\tau_t}^{(s)kT} e_{\tau_t}^{(s)k}\big]$$
$$+ k_1 \sum_{i=1}^{M} \big[e_i^{(s)T}(t) R_1 e_i^{(s)}(t) \exp(\mu \tau_0) - (1 - \varsigma) e_{i\tau_t}^{(s)T} R_1 e_{i\tau_t}^{(s)}\big]. \quad (15)$$

For convenience, we may as well let $e^{(s)}(t) = (e^{(s)1T}(t)\ e^{(s)2T}(t)\ \ldots\ e^{(s)nT}(t))^T$, then inequality (15) can be given as

$$
\begin{aligned}
&LV_1'(t, e^{(s)}(t)) \\
&\le \sum_{i=1}^{M} e_i^{(s)T}(t) A_{\mu_i} e_i^{(s)}(t) + \sum_{k=1}^{n} \beta e_{\tau_t}^{(s)kT} e_{\tau_t}^{(s)k} \\
&\quad + \sum_{k=1}^{n} \alpha e^{(s)kT}(t) e^{(s)k}(t) + \left(2p + \frac{\mu}{2}\right) e^{(s)T}(t) e^{(s)}(t) \\
&\quad + e^{(s)T}(t) \left\{ c\Gamma \otimes \left[\vartheta H + \frac{1}{2\theta} H H^T + \varepsilon^2 \theta \left(1 - \frac{1}{M}\right) I_M \right] \right\} \\
&\quad \times e^{(s)}(t) + 2q e_{\tau_t}^{(s)T} e_{\tau_t}^{(s)} + e^{(s)T}(t) \\
&\quad \times [k_1 \exp(\mu\tau_0) R_1 \otimes I_M] e^{(s)}(t) - e_{\tau_t}^{(s)T} [k_1 (1-\varsigma) R_1 \otimes I_M] e_{\tau_t}^{(s)} \\
&\le e^{(s)T}(t) \left\{ c\Gamma \otimes \left[\vartheta H + \frac{1}{2\theta} H H^T + \varepsilon^2 \theta \left(1 - \frac{1}{M}\right) I_M \right] \right. \\
&\qquad + \left(\alpha + 2p + \frac{\mu}{2}\right) I_n \otimes I_M \\
&\qquad \left. + k_1 \exp(\mu\tau_0) R_1 \otimes I_M \right\} e^{(s)}(t) \\
&\quad + e_{\tau_t}^{(s)T} \{ (\beta + 2q) I_n \otimes I_M - k_1 (1-\varsigma) R_1 \otimes I_M \} e_{\tau_t}^{(s)} \\
&\le e^{(s)T}(t) \left\{ c \left[\vartheta \lambda_2(H) + \frac{1}{2\theta} \lambda_{\max}(H H^T) + \varepsilon^2 \theta \left(1 - \frac{1}{M}\right) \right] I_n \right. \\
&\qquad \left. + \left(\alpha + 2p + \frac{\mu}{2}\right) I_n + k_1 \exp(\mu\tau_0) R_1 \right\} \otimes I_M e^{(s)}(t) \\
&\quad + e_{\tau_t}^{(s)T} \{ (\beta + 2q) I_n - k_1 (1-\varsigma) R_1 \} \otimes I_M e_{\tau_t}^{(s)}
\end{aligned} \tag{16}
$$

where $A_{\mu_i} < 0$, and $\{\alpha, \beta\} = \max\{\alpha_i, \beta_i | i = 1, 2, \ldots, M\}$. Let $\xi(t) = (e^{(s)T}(t)\ e_{\tau_t}^{(s)T})^T$ and

$$
\Xi = \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2^T & \Sigma_3 \end{pmatrix} \tag{17}
$$

where $\Sigma_1 = \{c[\vartheta\lambda_2(H) + (1/2\theta)\lambda_{\max}(HH^T) + \varepsilon^2\theta(1 - (1/M))]I_n + (\alpha + 2p + (\mu/2))I_n + k_1\exp(\mu\tau_0)R_1\} \otimes I_M < 0$, $\Sigma_3 = \{(\beta + 2q)I_n - k_1(1-\varsigma)R_1\} \otimes I_M < 0$, and $\Sigma_2 = \Sigma_2^T = \mathbf{0}$. According to the Lyapunov stability theory, it is easy to get that the error system (5) is asymptotically stable.

Taking the mathematical expectation on both sides of (8) and considering (16), we have

$$
\frac{dE\{V_1(t, e^{(s)}(t))\}}{dt} \le E\{\xi^T(t) \Xi \xi(t)\} \exp(\mu t). \tag{18}
$$

Note from (17) that $\Xi < 0$, one can further deduce that

$$
\frac{dE\{V_1(t, e^{(s)}(t))\}}{dt} \le -\lambda_{\min}(-\Xi) E\left\{ \left\| e^{(s)}(t) \right\|^2 \right\} \exp(\mu t). \tag{19}
$$

It follows from (18) that $E\{V_1(t, e^{(s)}(t))\} \le E\{V_1(0, e_0^{(s)})\}$ $(t \ge 0)$, which implies:

$$
\begin{aligned}
E\{\|e^{(s)}(t)\|^2\} &\le 2E\{V_1(t, e^{(s)}(t))\} \exp(-\mu t) \\
&\le 2E\left\{ V_1(0, e_0^{(s)}) \right\} \exp(-\mu t).
\end{aligned} \tag{20}
$$

Obviously, it can be found a positive scalar M_0, such that $2V_1(0, e_0^{(s)}) \le M_0$. That is to say, we have $E\{\|e^{(s)}(t)\|^2\} \le M_0 \exp(-\mu t)$. Thus, the leaders in the global leaders' network are all synchronized to their corresponding reference states with the exponential rate of convergence, and this completes the proof. ∎

Through Theorem 1, it is easy to observe that $e_i^{(sk)}(t)$ exponentially approach to zero, and thus, the dynamics of the reference states (4) can be equal to the following form:

$$
\begin{aligned}
d\bar{s}_k(t) &= [A_k \bar{s}_k(t) + f_k(t, \bar{s}_k(t), \bar{s}_k(t - \tau_t))] dt \\
&\quad + \delta(t, \bar{s}_k(t), \bar{s}_k(t - \tau_t)) d\omega + O(\exp(-\mu t)) dt \\
&\qquad k = 1, \ldots, m
\end{aligned} \tag{21}
$$

where $O(\exp(-\mu t))$ is a high-order infinitesimal of the term $\exp(-\mu t)$. Let $e_i^{(k)}(t) = x_i^{(k)}(t) - \bar{s}_k(t)$ be the error states from the nodes in the kth followers' subnetwork to the average state of their leaders, where $i = r_{k-1} + 1, \ldots, r_k;\ k = 1, 2, \ldots, m$. Subtracting (2) from (21), one can get the following error dynamical system between the nodes in the kth followers' subnetwork and their reference state:

$$
\begin{aligned}
de_i^{(k)}(t) &= \big\{ A_k e_i^{(k)}(t) + f_k\big(t, x_i^{(k)}, x_{i\tau_t}^{(k)}\big) - f_k\big(t, \bar{s}_k, \bar{s}_{k\tau_t}\big) \\
&\quad + c \sum_{j=1}^{N} b_{ij}^{(k)} \Gamma[g(x_j(t)) - g(\bar{s}_k(t))] \\
&\quad + u_i^{(k)}(t) + O(\exp(-\mu t)) \big\} dt \\
&\quad + \big\{ \delta\big(t, x_i^{(k)}, x_{i\tau_t}^{(k)}\big) - \delta\big(t, \bar{s}_k, \bar{s}_{k\tau_t}\big) \big\} d\omega \\
&\qquad i = r_{k-1} + 1, \ldots, l_k \\
de_i^{(k)}(t) &= \big\{ A_k e_i^{(k)}(t) + f_k\big(t, x_i^{(k)}, x_{i\tau_t}^{(k)}\big) - f_k(t, \bar{s}_k, \bar{s}_{k\tau_t}) \\
&\quad + c \sum_{j=1}^{N} b_{ij}^{(k)} \Gamma[g(x_j(t)) - g(\bar{s}_k(t))] \\
&\quad + O(\exp(-\mu t)) \big\} dt \\
&\quad + \big\{ \delta\big(t, x_i^{(k)}, x_{i\tau_t}^{(k)}\big) - \delta(t, \bar{s}_k, \bar{s}_{k\tau_t}) \big\} d\omega \\
&\qquad i = l_k + 1, \ldots, r_k.
\end{aligned} \tag{22}
$$

Theorem 2: Let Assumptions A1–A5 hold, the adaptive pinning controller $u_i^{(k)}(t)$ and the corresponding adaptive update law of feedback gains are given by

$$
\begin{aligned}
u_i^{(k)}(t) &= -c \sum_{j=w_{k-1}+1}^{w_k} d_{ij}^{(k)}(t) \Gamma\big[g(x_i^{(k)}(t)) - g(s_j(t)) \big] \\
\dot{d}_{ij}^{(k)}(t) &= c(\vartheta - \varepsilon) k_{ij} \big(x_i^{(k)} - s_j \big)^T \Gamma \big(x_i^{(k)} - s_j \big) \exp(\mu t) \\
&\qquad i = r_{k-1} + 1, \ldots, l_k; \quad j = w_{k-1} + 1, \ldots, w_k
\end{aligned} \tag{23}
$$

where $k = 1, 2, \ldots, m$. Then, the nodes in the controlled network (2) are exponentially synchronized to their corresponding reference state if there exist a positive-definite matrix $R_2 \in R^{n \times n}$ and a diagonal matrix $\bar{X} \in R^{N \times N}$, such that the following conditions hold:

$$
\begin{aligned}
Z_1 &= c \left[\vartheta B + \frac{1}{2\theta} B B^T \right] \\
&\quad + \left[c\varepsilon^2 \theta \left(1 - \frac{1}{N}\right) + \tilde{\alpha} + \frac{\tilde{p}}{2} + \frac{\mu}{2} \right. \\
&\quad \left. + k_2 \exp(\mu\tau_0) \lambda_{\max}(R_2) \right] I_N - c(\vartheta - \varepsilon) \bar{X} < 0 \\
Z_3 &= \left(\frac{\tilde{q}}{2} + \tilde{\beta} \right) I_n - k_2 (1 - \varsigma) R_2 < 0
\end{aligned} \tag{24}
$$

where μ, θ, and k_2 are the given positive constants, $\{\tilde{\alpha}, \tilde{\beta}, \tilde{p}, \tilde{q}\} = \max\{\alpha_i, \beta_i, p_i, q_i | i = 1, 2, \ldots, N\}$,

$$\bar{X} = \mathrm{diag}\{\underbrace{D_1^{*(1)}, \ldots, D_{l_1}^{*(1)}, 0, \ldots, 0}_{N_1}, \underbrace{D_{r_1+1}^{*(2)}, \ldots, D_{l_2}^{*(2)}, 0, \ldots, 0}_{N_2}, \ldots, \underbrace{D_{r_{m-1}+1}^{*(m)}, \ldots, D_{l_m}^{*(m)}, 0, \ldots, 0}_{N_m}\}$$

and $D_i^{*(k)}$ is the estimation of $D_i^{(k)}$.

Proof: From the detailed analysis of Theorem 1, we can get that all the leaders in each leaders' subnetwork (3) have exponentially synchronized to their corresponding reference state, where $k = 1, 2, \ldots, m$. Then, we consider the Lyapunov functional candidate for the error system (22)

$$\begin{aligned} V_2(t, e(t)) &= \frac{1}{2}\sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{r_k} e_i^{(k)T}(t)e_i^{(k)}(t)\exp(\mu t) \\ &\quad + \sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{l_k}\sum_{j=\omega_{k-1}+1}^{\omega_k} \frac{\left(d_{ij}^{(k)} - d_{ij}^{*(k)}\right)^2}{2k_{ij}} \\ &\quad + k_2\sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{r_k}\int_{t-\tau_t}^{t} e_i^{(k)T}(\upsilon)R_2 e_i^{(k)}(\upsilon) \\ &\quad \times\exp(\mu(\upsilon + \tau_t))d\upsilon. \end{aligned} \tag{25}$$

The stochastic differential $dV_2(t, e(t))$ along the trajectories of (22) gives

$$\begin{aligned} dV_2(t, e(t)) &= LV_2(t, e(t))dt \\ &\quad + \sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{r_k} e_i^{(k)T}(t)Y_i^{(k)}\exp(\mu t)d\omega \end{aligned} \tag{26}$$

where $Y_i^{(k)} = \delta(t, x_i^k(t), x_{i\tau_t}^k) - \delta(t, \bar{s}_k(t), \bar{s}_{k\tau_t})$.

For convenience, we may as well let $LV_2(t, e(t)) = LV_2'(t, e(t))\exp(\mu t)$, where

$$\begin{aligned} &LV_2'(t, e(t)) \\ &= \sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{r_k} e_i^{(k)T}(t)A_k e_i^{(k)}(t) \\ &\quad + \sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{r_k} e_i^{(k)T}(t)\left[f_k\left(t, x_i^{(k)}, x_{i\tau_t}^{(k)}\right) - f_k(t, \bar{s}_k, \bar{s}_{k\tau_t})\right] \\ &\quad + \sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{r_k} ce_i^{(k)T}(t)\sum_{j=1}^{N} b_{ij}^{(k)}\Gamma[g(x_j) - g(\bar{s}_k)] \\ &\quad - \sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{l_k} ce_i^{(k)T}(t)\sum_{j=\omega_{k-1}+1}^{\omega_k} d_{ij}^{(k)}\Gamma\left[g\left(x_i^{(k)}\right) - g(s_j)\right] \\ &\quad + \frac{1}{2}\sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{r_k}\mathrm{trace}\left(Y_i^{(k)T}Y_i^{(k)}\right) \\ &\quad + \frac{\mu}{2}\sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{r_k} e_i^{(k)T}(t)e_i^{(k)}(t) \\ &\quad + \sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{l_k}\sum_{j=\omega_{k-1}+1}^{\omega_k} c(\vartheta - \varepsilon)\left(d_{ij}^{(k)} - d_{ij}^{*(k)}\right)e_i^{(k)T}\Gamma e_i^{(k)} \\ &\quad + k_2\sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{r_k} e_i^{(k)T}(t)R_2 e_i^{(k)}(t)\exp(\mu\tau_t) \\ &\quad - k_2\sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{r_k}(1 - \dot{\tau}_t)e_{i\tau_t}^{(k)T}R_2 e_{i\tau_t}^{(k)}. \end{aligned} \tag{27}$$

In view of Assumption A2, one can obtain

$$\begin{aligned} LV_2'(t, e(t)) &\le \sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{r_k} e_i^{(k)T}(t)A_k e_i^{(k)}(t) \\ &\quad + \sum_{i=1}^{N}\alpha_i e_i^T(t)e_i(t) + \sum_{i=1}^{N}\beta_i e_{i\tau_t}^T e_{i\tau_t} + V_2 + V_3 \\ &\quad + c\sum_{k=1}^{n}\gamma_k e^{kT}(t) \\ &\quad \times\left[\vartheta B + \frac{1}{2\theta}BB^T + \varepsilon^2\theta\left(1 - \frac{1}{N}\right)I_N\right]e^k(t) \end{aligned} \tag{28}$$

where $e^k(t) = (e_{1k}(t)\ e_{2k}(t)\ \ldots\ e_{Nk}(t))^T$. Note that $g(\cdot) \in \mathrm{NCF}(\vartheta, \varepsilon)$, we can get

$$\begin{aligned} V_2 &= -\sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{l_k} ce_i^{(k)T}(t)\sum_{j=\omega_{k-1}+1}^{\omega_k} d_{ij}^{(k)}\Gamma\left[g\left(x_i^{(k)}\right) - g(s_j)\right] \\ &= -\sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{l_k} c\sum_{j=1}^{n}\gamma_j e_{ij}^{(k)}(t)D_i^{(k)}\left[g\left(x_{ij}^{(k)}\right) - g(\bar{s}_{\mu_i j})\right] \\ &\le -c(\vartheta - \varepsilon)\sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{l_k}\sum_{j=1}^{n}\gamma_j D_i^{(k)}e_{ij}^{(k)}(t)e_{ij}^{(k)}(t) \\ &= -c(\vartheta - \varepsilon)\sum_{k=1}^{n}\gamma_k e^{kT}(t)De^k(t) \end{aligned} \tag{29}$$

where $D_i^{(k)} = \sum_{j=\omega_{k-1}+1}^{\omega_k} d_{ij}^{(k)}$ and

$$D = \mathrm{diag}\{\underbrace{D_1^{(1)}, \ldots, D_{l_1}^{(1)}, 0, \ldots, 0}_{N_1}, \underbrace{D_{r_1+1}^{(2)}, \ldots, D_{l_2}^{(2)}, 0, \ldots, 0}_{N_2}, \ldots, \underbrace{D_{r_{m-1}+1}^{(m)}, \ldots, D_{l_m}^{(m)}, 0, \ldots, 0}_{N_m}\}.$$

In view of Assumptions A3 and A5, we have

$$\begin{aligned} V_3 &= \sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{l_k}\sum_{j=\omega_{k-1}+1}^{\omega_k} c(\vartheta - \varepsilon)\left(d_{ij}^{(k)} - d_{ij}^{*(k)}\right)e_i^{(k)T}\Gamma e_i^{(k)} \\ &\quad + \frac{1}{2}\sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{r_k}\mathrm{trace}\left(Y_i^{(k)T}Y_i^{(k)}\right) + \frac{\mu}{2}\sum_{i=1}^{N} e_i^T(t)e_i(t) \\ &\quad + k_2\sum_{i=1}^{N}\left[e_i^T(t)R_2 e_i(t)\exp(\mu\tau_t) - (1 - \dot{\tau}_t)e_{i\tau_t}^T R_2 e_{i\tau_t}\right] \\ &\le c(\vartheta - \varepsilon)\sum_{k=1}^{m}\sum_{i=r_{k-1}+1}^{l_k} e_i^{(k)T}(t)\left(D_i^{(k)} - D_i^{*(k)}\right)\Gamma e_i^{(k)}(t) \\ &\quad + \sum_{k=1}^{n}\left[\frac{\tilde{p}}{2}e^{kT}(t)e^k(t) + \frac{\tilde{q}}{2}e_{\tau_t}^{kT}e_{\tau_t}^k\right] + \frac{\mu}{2}\sum_{k=1}^{n} e^{kT}(t)e^k(t) \\ &\quad + k_2\sum_{i=1}^{N}\left[e_i^T(t)R_2 e_i(t)\exp(\mu\tau_0) - (1 - \varsigma)e_{i\tau_t}^T R_2 e_{i\tau_t}\right] \end{aligned} \tag{30}$$

where $\{\tilde{p}, \tilde{q}\} = \max\{p_i, q_i | i = 1, 2, \ldots, N\}$, and $D_i^{*(k)} = \sum_{j=\omega_{k-1}+1}^{\omega_k} d_{ij}^{*(k)}$. By substituting inequalities (29) and (30) into (28), one can obtain that

$$
\begin{aligned}
&LV_2'(t, e(t)) \\
&\leq \sum_{i=1}^{N} e_i^T(t) A_{\mu_i} e_i(t) + \sum_{k=1}^{n} \tilde{\alpha} e^{kT}(t) e^k(t) \\
&\quad + \sum_{k=1}^{n} \tilde{\beta} e_{\tau_t}^{kT} e_{\tau_t}^k - c(\vartheta - \varepsilon) \sum_{k=1}^{n} \gamma_k e^{kT}(t) D e^k(t) \\
&\quad + c \sum_{k=1}^{n} \gamma_k e^{kT}(t) \left[\vartheta B + \frac{1}{2\theta} B B^T + \varepsilon^2 \theta \left(1 - \frac{1}{N}\right) I_N\right] e^k(t) \\
&\quad + c(\vartheta - \varepsilon) \sum_{k=1}^{n} \gamma_k e^{kT}(t) (D - \bar{X}) e^k(t) \\
&\quad + \sum_{k=1}^{n} \left[\frac{\tilde{p}}{2} e^{kT}(t) e^k(t) + \frac{\tilde{q}}{2} e_{\tau_t}^{kT} e_{\tau_t}^k\right] + \frac{\mu}{2} \sum_{k=1}^{n} e^{kT}(t) e^k(t) \\
&\quad + k_2 \sum_{i=1}^{N} \left[e_i^T(t) R_2 e_i(t) \exp(\mu \tau_0) - (1 - \varsigma) e_{i\tau_t}^T R_2 e_{i\tau_t}\right]
\end{aligned} \tag{31}
$$

where $\{\tilde{\alpha}, \tilde{\beta}\} = \max\{\alpha_i, \beta_i | i = 1, 2, \ldots, N\}$, and $\bar{X} = \mathrm{diag}\{D_1^{*(1)}, \ldots, D_{l_1}^{*(1)}, 0, \ldots, 0, D_{r_1+1}^{*(2)}, \ldots, D_{l_2}^{*(2)}, 0, \ldots, 0, \ldots, D_{r_{m-1}+1}^{*(m)}, \ldots, D_{l_m}^{*(m)}, 0, \ldots, 0\}$. As the fact that $A_{\mu_i} < 0$, and let $e(t) = (e^{1T}(t)\ e^{2T}(t)\ \ldots\ e^{nT}(t))^T$, then inequality (31) can be given as

$$
\begin{aligned}
&LV_2'(t, e(t)) \\
&\leq \sum_{k=1}^{n} \tilde{\alpha} e^{kT}(t) e^k(t) + \sum_{k=1}^{n} \tilde{\beta} e_{\tau_t}^{kT} e_{\tau_t}^k \\
&\quad + c \sum_{k=1}^{n} \gamma_k e^{kT}(t) \left[\vartheta B + \frac{1}{2\theta} B B^T + \varepsilon^2 \theta \left(1 - \frac{1}{N}\right) I_N\right] \\
&\quad \times e^k(t) - c(\vartheta - \varepsilon) \sum_{k=1}^{n} \gamma_k e^{kT}(t) \bar{X} e^k(t) \\
&\quad + \sum_{k=1}^{n} \left[\frac{\tilde{p}}{2} e^{kT}(t) e^k(t) + \frac{\tilde{q}}{2} e_{\tau_t}^{kT} e_{\tau_t}^k\right] + \frac{\mu}{2} \sum_{k=1}^{n} e^{kT}(t) e^k(t) \\
&\quad + k_2 \sum_{i=1}^{N} \left[e_i^T(t) R_2 e_i(t) \exp(\mu \tau_0) - (1 - \varsigma) e_{i\tau_t}^T R_2 e_{i\tau_t}\right] \\
&\leq e^T(t) \Bigg\{ c\Gamma \otimes \left[\vartheta B + \frac{1}{2\theta} B B^T + \varepsilon^2 \theta \left(1 - \frac{1}{N}\right) I_N\right] \\
&\qquad + \left(\tilde{\alpha} + \frac{\tilde{p}}{2} + \frac{\mu}{2}\right) \Gamma \otimes I_N + k_2 \exp(\mu \tau_0) \lambda_{\max}(R_2) \Gamma \otimes I_N \\
&\qquad - c(\vartheta - \varepsilon) \Gamma \otimes \bar{X} \Bigg\} e(t) \\
&\quad + e_{\tau_t}^T \left\{ \left(\frac{\tilde{q}}{2} + \tilde{\beta}\right) I_n \otimes I_N - k_2(1 - \varsigma) R_2 \otimes I_N \right\} e_{\tau_t} \\
&= \zeta^T \Xi' \zeta
\end{aligned} \tag{32}
$$

where $\zeta(t) = (e^T(t)\ e_{\tau_t}^T)^T$, and $\Xi' = \begin{bmatrix} \widehat{Z}_1 & Z_2 \\ Z_2^T & Z_3 \end{bmatrix}$ is symmetrical and negative definite, in which $\widehat{Z}_1 = \Gamma \otimes \{c[\vartheta B + (1/2\theta) B B^T + \varepsilon^2 \theta (1 - (1/N)) I_N] + (\tilde{\alpha} + (\tilde{p}/2) + (\mu/2)) I_N + k_2 \exp(\mu \tau_0) \lambda_{\max}(R_2) I_N\} - c(\vartheta - \varepsilon) \Gamma \otimes \bar{X} = \Gamma \otimes Z_1 < 0$, $Z_3 = ((\tilde{q}/2) + \tilde{\beta}) I_n \otimes I_N - k_2(1 - \varsigma) R_2 \otimes I_N < 0$, and $Z_2 = Z_2^T = \mathbf{0}$. The rest procedure of the proof is the same as that in Theorem 1, and hence, we omit it here. This completes the proof. ■

From condition (24), we can easily get that the nodes in the followers' subnetwork (2) are all exponentially synchronized to their reference state under the given adaptive feedback pinning controller (23).

Remark 4: From conditions (6) and (24), we can see that the larger the coupling strength is, the easier these conditions are satisfied. Furthermore, our network model is the generalized form of the models proposed in the previous works [20]–[22], and it can be applied to not only the complex networks with identical node but also the multiple subnetworks of complex networks with nonidentical ones. In particular, if $M \equiv 1$ and $m \equiv 1$, then the present framework is the original complete synchronization of complex networks with identical node; if $M_k \equiv 1$ and $k = 1, 2, \ldots, m$, and there is no connection between the leaders, then the present framework becomes the traditional cluster synchronization of complex networks with nonidentical nodes; specially, if there is only one leaders' network and all the nodes in each followers' subnetwork are assumed to be identical, then the framework is the same as that in [20].

Remark 5: As the fact that $\Gamma \geq I_n$, condition (6) in Theorem 1 can be simplified as follows:

$$
\begin{aligned}
\Sigma_1 &= c\left[\vartheta \lambda_2(H) + \frac{1}{2\theta} \lambda_{\max}(H H^T) + \varepsilon^2 \theta \left(1 - \frac{1}{M}\right)\right] \\
&\quad + \alpha + 2p + \frac{\mu}{2} + k_1 \exp(\mu \tau_0) \lambda_{\max}(R_1) < 0 \\
\Sigma_3 &= \beta + 2q - k_1(1 - \varsigma) \lambda_{\min}(R_1) < 0
\end{aligned} \tag{33}
$$

and condition (24) in Theorem 2 can be simplified as

$$
\begin{aligned}
Z_1 &= \Bigg\{c\left[\vartheta \lambda_2(B) + \frac{1}{2\theta} \lambda_{\max}(B B^T) + \varepsilon^2 \theta \left(1 - \frac{1}{N}\right)\right] + \tilde{\alpha} \\
&\quad + \frac{\tilde{p}}{2} + \frac{\mu}{2} + k_2 \exp(\mu \tau_0) \lambda_{\max}(R_2)\Bigg\} I_N - c(\vartheta - \varepsilon) \bar{X} < 0 \\
Z_3 &= \left[\frac{\tilde{q}}{2} + \tilde{\beta} - k_2(1 - \varsigma) \lambda_{\min}(R_2)\right] I_n < 0.
\end{aligned} \tag{34}
$$

From inequalities (33) and (34), it is easy to get that the topology structure of each subnetwork plays an important role in realizing the cluster synchronization on multiple subnetworks of complex networks. Inequalities (33) and (34) are very similar, but there is a negative feedback item $-c(\vartheta - \varepsilon)\bar{X}$ in (34), such that condition (34) is easier to be satisfied than (33). Only when these conditions in Theorems 1 and 2 are satisfied simultaneously, we can say that all the nodes have exponentially synchronized to their corresponding reference states.

Remark 6: Many similar pinning control results have been given in the previous works, but most of them have focused on the pinning synchronization of a single network or two networks; while in this paper, the proposed pinning scheme can be applied to not only a single network but also the multiple subnetworks of complex networks with nonidentical nodes. In particular, when the coupling function is linear, we can get Corollary 1.

Corollary 1: If the nonlinear coupling function $g(x)$ is assumed to be linear, e.g., $g(x) = x$, then we can get the following conditions for the global leaders' network:

$$\Sigma_1 = c\lambda_2(H) + \alpha + 2p + \frac{\mu}{2} + k_1\exp(\mu\tau_0)\lambda_{\max}(R_1) < 0$$
$$\Sigma_3 = \beta + 2q - k_1(1-\varsigma)\lambda_{\min}(R_1) < 0 \quad (35)$$

and the ones for the global followers' network

$$Z_1 = \left[c\lambda_2(B) + \tilde{\alpha} + \frac{\tilde{p}}{2} + \frac{\mu}{2} + k_2\exp(\mu\tau_0)\lambda_{\max}(R_2)\right] I_N - c\bar{X} < 0$$
$$Z_3 = \left[\frac{\tilde{q}}{2} + \tilde{\beta} - k_2(1-\varsigma)\lambda_{\min}(R_2)\right] I_n < 0 \quad (36)$$

where the adaptive pinning controller $u_i^{(k)}(t)$ and the corresponding adaptive update law can be given by

$$u_i^{(k)}(t) = -c\sum_{j=w_{k-1}+1}^{w_k} d_{ij}^{(k)}(t)\Gamma\left(x_i^{(k)}(t) - s_j(t)\right)$$
$$\dot{d}_{ij}^{(k)}(t) = ck_{ij}\left(x_i^{(k)} - s_j\right)^T\Gamma\left(x_i^{(k)} - s_j\right)\exp(\mu t)$$
$$i = r_{k-1}+1,\ldots,l_k;\quad j = w_{k-1}+1,\ldots,w_k. \quad (37)$$

Clearly, conditions (35) and (36) are very similar with the conditions presented in [20]. The main difference is that the conditions considered in [20] are very ideal, and only the connections of the nodes within the same subnetwork are considered, and the synchronous mode of the nodes is complete synchronization; while in this paper, we consider not only the influence of stochastic factors but also the cooperation and competition of the nodes between different subnetworks, and the synchronous pattern is cluster synchronization. Moreover, the feedback gains of the controllers presented in [20] are much larger than the required values, which is unpractical in the reality; while the adaptive update law in (37) can deal with this problem very well.

B. Design of Pinning Control Scheme and the Selection Scheme of Pinned Nodes for the Multiple Nonlinearly Coupled Dynamical Subnetworks of Complex Networks

In Section III-A, some cluster synchronization criteria on multiple nonlinearly coupled dynamical subnetworks of complex networks have been given. Next, we will make a brief analysis to condition (24) in Theorem 2 to determine the minimum number of nodes that need to be controlled, and give the detailed scheme on the selection of the pinned nodes. As we know that the global followers' network consists of m followers' subnetworks, in order to obtain the number of pinned nodes for each subnetwork, we will analyze these subnetworks one by one. The matrix B can be represented as $B = [B^{(1)T}\ B^{(2)T}\ \ldots\ B^{(m)T}]^T$, where $B^{(k)} = (B_{k1}\ B_{k2}\ \ldots\ B_{km})$, $B_{kj} \in R^{N_k\times N_j}$, and $k, j = 1, 2, \ldots, m$. Note that the sum of the energy transmission between two different subnetworks is zero, the influence of the coupling between two different subnetworks is far weaker than that within the same subnetwork. In view of this, according to the characteristic of the matrix B and condition (24), for the kth subnetwork, we can get

$$\tilde{Z}_k = c\vartheta B_{kk} + L_k - c(\vartheta-\varepsilon)\bar{X}_k = \begin{pmatrix} \tilde{A}_k - \tilde{D}_k & \tilde{B}_k \\ \tilde{B}_k^T & \tilde{C}_k \end{pmatrix} \quad (38)$$

where $L_k = \gamma_k I_{N_k}$, $\gamma_k = (c/2\theta)\lambda_{\max}(B^{(k)}B^{(k)T}) + c\varepsilon^2\theta(1-(1/N_k)) + \tilde{\alpha}_k + (\tilde{p}_k/2) + (\mu/2) + k_2\exp(\mu\tau_0)\lambda_{\max}(R_2)$, $\bar{X}_k = \mathrm{diag}\{D_{r_{k-1}+1}^{*(k)}, \ldots, D_{l_k}^{*(k)}, 0, \ldots, 0\}$, $\tilde{A}_k$ and $\tilde{B}_k$ are the matrices with appropriate dimensions, $\tilde{D}_k = c(\vartheta-\varepsilon)\,\mathrm{diag}\{D_{r_{k-1}+1}^{*(k)}, \ldots, D_{l_k}^{*(k)}\}$, and $\tilde{C}_k$ is obtained by moving the first l_k row–column pairs of matrix $\tilde{Z}_k$.

Assume that the pinned nodes in the kth followers' subnetwork are just the first l_k nodes, if not, we can exchange the position of the nodes, such that the nodes in need of pinning are transformed into the first l_k nodes. Thus, the connections between all the pinned nodes are equivalent to $\tilde{A}_k - \tilde{D}_k$, and the connections between the pinned nodes and the unpinned nodes are equivalent to $\tilde{B}_k$ or $\tilde{B}_k^T$, while the connections between the unpinned nodes are equivalent to $\tilde{C}_k$. According to the constraint of the matrix B and the expression of $\tilde{Z}_k$, we can get that all of the elements in the matrix $\tilde{B}_k$ are positive, and the diagonal elements of the matrix $\tilde{C}_k$ are negative, while all the nondiagonal elements of which are positive. In view of Lemma 1, we can easily get that $\tilde{Z}_k < 0$ is equivalent to $\tilde{C}_k < 0$ because of $\tilde{A}_k - \tilde{D}_k - \tilde{B}_k\tilde{C}_k^{-1}\tilde{B}_k^T < 0$ by choosing $\tilde{D}_k > \lambda_{\max}(\tilde{A}_k - \tilde{B}_k\tilde{C}_k^{-1}\tilde{B}_k^T)I_{l_k}$. Note that $\tilde{C}_k = (\tilde{Z}_k)_{[l_k]} = c\vartheta(B_{kk})_{[l_k]} + \gamma_k I_{N_k-l_k}$, where $(\tilde{Z}_k)_{[l_k]}$ represents the minor matrix of $\tilde{Z}_k$ by removing its first l_k row–column pairs. Therefore, with choosing large enough $\tilde{D}_k$, we have $c\vartheta(B_{kk})_{[l_k]} + \gamma_k I_{N_k-l_k} < 0$. Then, one can immediately get Corollary 2.

Corollary 2: If there exists a l_k, such that $\lambda_{\max}(B_{kk})_{[l_k]} < -(\gamma_k/c\vartheta)$ for the kth followers' subnetwork, then the nodes in system (2) are globally exponentially synchronized to their corresponding reference state under the given adaptive pinning controllers, where $k = 1, 2, \ldots, m$.

Next, we will give the selection scheme of the pinned nodes. In view of $Z_1 < 0$ and the fact that the main diagonal elements of a negative definite and symmetrical matrix are negative, we have $(Z_1)_{ii} < 0, i = 1, 2, \ldots, N$, which can be described as

$$\begin{cases} (Z_1)_{ii} = c\vartheta b_{ii} + \gamma - c(\vartheta-\varepsilon)D_i^{*(\mu_i)} < 0, & i \in \chi \\ (Z_1)_{ii} = c\vartheta b_{ii} + \gamma < 0, & i \in C - \chi \end{cases} \quad (39)$$

where $\gamma = (c/2\theta)\lambda_{\max}(BB^T) + c\varepsilon^2\theta(1-(1/N)) + \tilde{\alpha} + (\tilde{p}/2) + (\mu/2) + k_2\exp(\mu\tau_0)\lambda_{\max}(R_2)$, χ represents a node-set that consists of all the pinned nodes in the global followers' network, and $C = C_1 \cup C_2 \cup \ldots \cup C_m$, accordingly, $C - \chi$ is a node-set that consists of all the unpinned nodes in the global followers' network. In view of $b_{ii} < 0$ and the fact that $\mathrm{Deg}(i) = -b_{ii}$, from the second formula in (39), we can get that $\mathrm{Deg}(i) > (\gamma/c\vartheta)$, where $i \in C - \chi$. That is to say, the nodes with low degrees need to be controlled first. On the other hand, it is easy to know that $(\tilde{B}_k^T \;\; \tilde{C}_k)$ is a γ_k-row-sum matrix. Therefore, if there are more connections in the matrix $\tilde{B}_k^T$, the matrix $\tilde{C}_k$ will tend to be negative, and $\tilde{C}_k < 0$, which means that the nodes with large degrees should be controlled, since these nodes can affect many connected nodes. As stated above, we can get that the nodes with very large or low degrees are good candidates for applying pinning controllers.

The specific pinning scheme on the selection of pinned nodes can be stated as follows.

1) Rearrange the network nodes: the nodes are followed by the other ones in descending order based on their degrees in each followers' subnetwork.
2) According to the condition $\lambda_{\max}(B_{kk})_{[l_k]} < -(\gamma_k/c\vartheta)$, we can get the least number of the pinned nodes for the kth followers' subnetwork. Moreover, due to the energy transmission is conserved between any two different subnetworks, at least two nodes in each followers' subnetwork need to be controlled. That is to say, the number of the pinned nodes for the kth followers' subnetwork is $\max\{2, l_k\}$.
3) The selection scheme for the pinned nodes is from both the left and the right to the middle in order at the same time, and the selection of priority for the nodes on the relative position depends on the specific network structure. If the degrees of the nodes are the same, we select them in turn.

Remark 7: The selection of priority for the pinned nodes on the relative position depends on the specific network structure. There are many similar works on the selection of pinned nodes and the calculation method for the number of pinned nodes [16], [30], [37], [38]. But most of these works have focused on the complex networks with identical node and without considering the influence of stochastic factors, or having all the nodes synchronized to an identical node, or the pinning synchronization scheme is confined to only one network or two networks, which is not very consistent with the actual. Moreover, the least number of pinned nodes obtained in theory is usually much larger than that required in practice [39]. Therefore, we just give a rough calculation scheme for the selection on the number of pinned nodes, while specific problem should be concrete analysis.

IV. Numerical Simulations

In this section, some examples are offered to illustrate the effectiveness of the theoretical analysis presented in Section III. Consider the general complex networks that consist of three pairs of different subnetworks, in which three different dynamical systems, such as CNN's neuron system [40], Hindmarsh-Rose neural system [41], and Lu neural oscillator [42], are selected as the corresponding node dynamics.

Case 1 (First Pair of Matching Subnetworks): The first pair of matching subnetworks consists of four followers and three leaders, in which CNN's neuron system [40] is taken as the corresponding node dynamics, and only the first two nodes in the followers' subnetwork need to be controlled. The CNN's neuron system can described as

$$\frac{dx}{dt} = -\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A_1}\underbrace{\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} p_1\tilde{g}(x_1) - s\tilde{g}(y_1) - s\tilde{g}(z_1) \\ -s\tilde{g}(x_1) + p_2\tilde{g}(y_1) - r\tilde{g}(z_1) \\ -s\tilde{g}(x_1) + r\tilde{g}(y_1) + p_3\tilde{g}(z_1) \end{pmatrix}}_{f_1(x)} \tag{40}$$

where $x = (x_1, y_1, z_1)^T \in R^3$, $\tilde{g}(x) = (1/2)(|x+1| - |x-1|)$, $p_1 = 1.25$, $p_2 = 1.1$, $p_3 = 1$, $s = 3.2$, and $r = 4.4$.

Case 2 (Second Pair of Matching Subnetworks): The second pair of matching subnetworks is composed of three followers and two leaders with Hindmarsh-Rose neural system [41] as the node dynamics, and only the first two nodes in the followers' subnetwork are in need of applying controllers, where the Hindmarsh-Rose neural system is described by

$$\frac{dx}{dt} = -\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A_2}\underbrace{\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} y_2 - x_2^3 + 3x_2^2 + x_2 - z_2 + 3.25 \\ 1 - 5x_2^2 \\ 0.02x_2 + 0.032 + 0.995z_2 \end{pmatrix}}_{f_2(x)}. \tag{41}$$

Case 3 (Third Pair of Matching Subnetworks): The third pair of matching subnetworks, with Lu neural oscillator [42] as the node dynamics, consists of four followers and three leaders, and only the first two nodes in the followers' subnetwork need to be controlled. The Lu neural oscillator can be described as

$$\dot{x}(t) = -Cx(t) + Af(x(t)) \tag{42}$$

where $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 3 & 5 \\ 0.1 & 2 \end{pmatrix}$, and $f(x(t)) = \tanh(x(t))$.

In the simulation process for the cluster synchronization of three pairs of subnetworks with different dynamical behaviors, we assume that $H^{(k)}$, $B_{kk}^{(k)} (k = 1, 2, 3)$ are all fully connected, and the crossover matrices can be taken as

$$B_{12}^{(1)} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_{13}^{(1)} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B_{23}^{(2)} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{21}^{(2)T} = B_{12}^{(1)}, B_{31}^{(3)T} = B_{13}^{(1)}$$

and $B_{32}^{(3)T} = B_{23}^{(2)}$. The other related parameters are selected as $p = q = 0.1$, $\alpha = \beta = 1.5$, $\mu = 1$, $\tau_t = e^t/6(1+e^t)$, $c = 4$, $\theta = 2$, and $\delta(t, x_i, x_{i\tau_t}) = 0.05(x_i - x_{i\tau_t})(i = 1, 2, \ldots, 11)$. The corresponding nonlinear coupling functions are selected as $g_1(x) = 10x + \sin(2x)$, $g_2(x) = 10x + \cos x$, $g_3(x) = 4x + 0.5\sin x$. The states of errors between the nodes in the

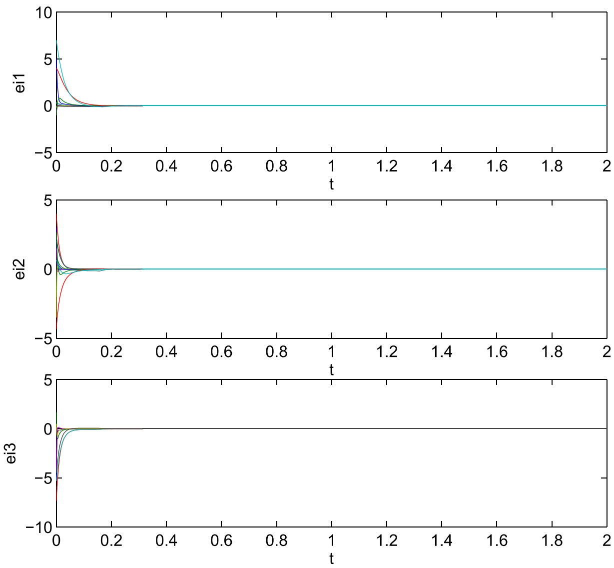

Fig. 2. States of errors between the nodes $x_i^{(\mu_i)} (i = 1, \ldots, 11)$ and their corresponding reference state $\bar{s}^{(\mu_i)}$ in the global followers' network.

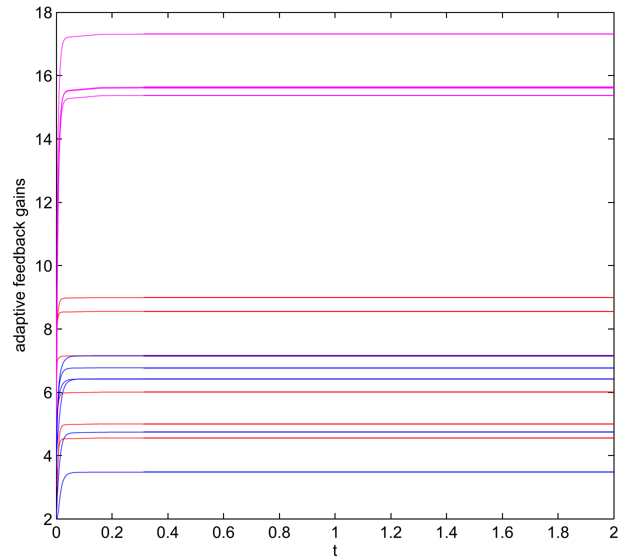

Fig. 3. Time evolution of adaptive feedback gains between the pinned nodes and their leaders, where the red lines represent that in the first pair of subnetwork, the mauve lines represent that in the second pair of subnetwork, and the blue lines represent that in the third pair of subnetwork.

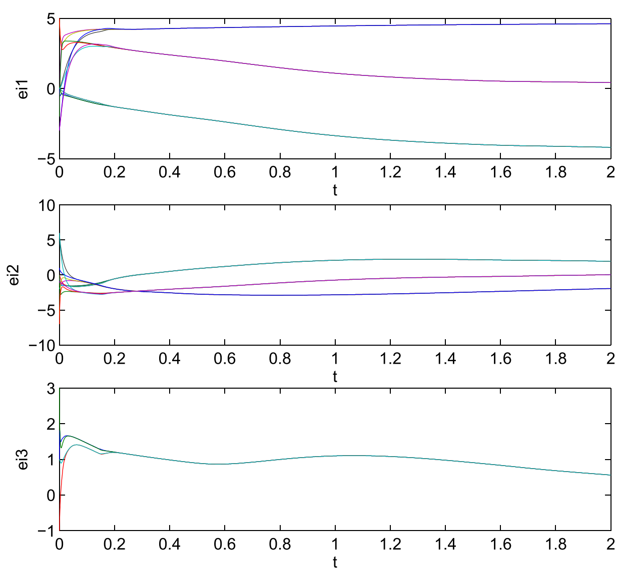

Fig. 4. Errors of the nodes between different followers' subnetworks.

global followers' network and their corresponding reference states are shown in Fig. 2. From the illustration of Fig. 2, one can get that all the error states converge to zero very quickly, and the adaptive feedback gains between the pinned nodes and their leaders all tend to some constants in a very short time, just as shown in Fig. 3. That is to say, the realization of the cluster synchronization on multiple nonlinearly coupled dynamical subnetworks of complex networks has good immunity to the influence of random factors and unknown feedback gains. Moreover, the errors of the nodes between different followers' subnetworks are shown in Fig. 4. From Fig. 4, we can see that the nodes belonging to different subnetworks cannot realize the synchronization at all, while as the fact that the nodes belonging to the same subnetwork have synchronized, accordingly, there are three clusters of asymptotic curves in the first two subfigures; as for the third subfigure, because the nodes in the third pair of subnetworks are 2-D, there is only a cluster of asymptotic curve that represents errors of the nodes between the first and second followers' subnetwork.

Remark 8: In this paper, the topology connections of the nodes in each subnetwork are assumed to be fully connected or tree-based, and according to the given pinning scheme on the selection of pinned nodes, we can just select the first two nodes as the nodes in need of applying controllers. Therefore, the selection of the pinned nodes for the numerical simulation is consistent with the given pinning scheme.

In addition, most of the previous works have just focused on the pinning synchronization of complex networks, while Lu *et al.* [20] realize the pinning synchronization on complex networks of networks with multiple leaders and followers. But the scheme proposed in [20] cannot be extended to the complex networks with nonidentical nodes, and only the connections of the nodes belonging to the same subnetwork are considered, which do not often accord with the actual. In fact, the nodes belonging to different subnetworks usually have nonidentical node dynamics, and they may also need to communicate with each other in a cooperative or even competitive way. However, these problems can be well resolved with the proposed scheme in this paper, and then, we will make a simple comparison on them. For convenience, we just consider the complex networks consisting of five leaders and eight followers, in which the leaders' network is tree-based and the followers' network is fully connected, and only the first two nodes need to be controlled. From the illustrations of Figs. 5 and 6, we can easily get that, compared with the scheme proposed in [20], our scheme can realize the cluster synchronization of all the nodes with less time and converge to zero very quickly even under the influence of stochastic disturbances and time-varying delays. More importantly, this

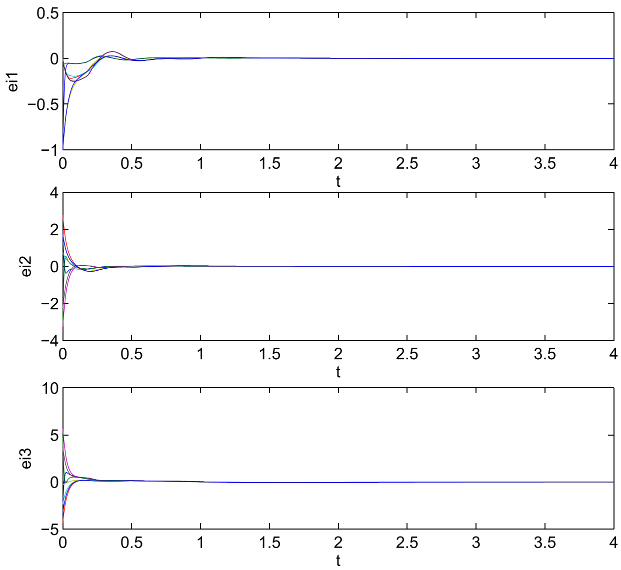

Fig. 5. States of errors between the nodes $x_i (i = 1, \ldots, 8)$ and their corresponding average state $\bar{s}^{(\mu_i)}$ in the global followers' network with tree-based connection by using our method.

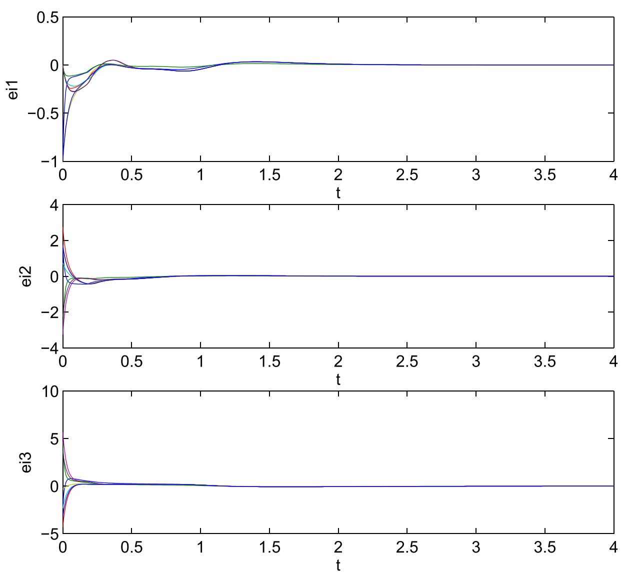

Fig. 6. States of errors between the nodes $x_i (i = 1, \ldots, 8)$ and their corresponding average state $\bar{s}^{(\mu_i)}$ in the global followers' network with tree-based connection by using the method proposed in [20].

improved network model is much more general and realistic, and it will have much more practical application in the near future.

V. Conclusion

In this paper, cluster synchronization problem on multiple nonlinearly coupled dynamical subnetworks of complex networks with nonidentical nodes and stochastic disturbances is investigated. Based on the general leader–follower's model, an improved network structure model that consists of multiple pairs of matching subnetworks is proposed, where each pair of matching subnetworks is composed of a leaders' subnetwork and a followers' subnetwork. The dynamics of the nodes in each pair of matching subnetworks are identical, while the ones belonging to different pairs of unmatched subnetworks are usually nonidentical, and some random factors are taken into consideration as well. There are many leaders in each leaders' subnetwork, which provides more opportunities for these leaders to transmit information, and the average state of these leaders is selected as the reference state. Furthermore, the leaders are only responsible for the information transmission, such as the necessary priori knowledge or professional skills, to their matching followers, but the completion of the task needs all the followers to cooperate with each other. Then, some cluster synchronization criteria are derived for both the global leaders' network and followers' network, and a suitable pinning control scheme that the nodes with very large or low degrees are good candidates for applying pinning control is given. Numerical simulations are proposed to validate the feasibility and effectiveness of the theoretical results.

In addition, though the proposed network model in this paper is simple, the theoretical results in this paper can provide some new insights for the possible applications in various emerging fields, such as artificial neural networks, intelligent system, multiagent systems, machine learning, and so on. However, there are still some valuable problems to be solved, which will be focused on in our future work, such as how to describe a cluster based on the topological characteristics, the selection of classification rules, finite-time control, nonlinear time-delayed coupling, and some application of the proposed scheme on all kinds of research fields, such as brain science and multiagent systems.

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