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Hybrid combinatorial synchronization on multiple sub-networks of complex network with unknown boundaries of uncertainties



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ABSTRACT

In this paper, hybrid combinatorial synchronization on multiple sub-networks of complex network with unknown boundaries of uncertainties, stochastic disturbances and different dimensions of nodes is investigated. The complex network can be divided into multiple sub-networks, such that the nodes belonging to the same sub-network are identical, while the ones belonging to different sub-networks are non-identical. Each sub-network is composed of two types of nodes: general node and preset-node, and only the nodes with the same type can communicate with each other. In this new setting, the aim is to design some suitable controllers on the preset-nodes, which are taken as the being synchronized nodes in each sub-network, so as to reach not only the combinatorial inner synchronization within a sub-network, but also the combinatorial outer synchronization between multiple different sub-networks. Furthermore, the nodes may be influenced by some stochastic disturbances and uncertainties, and the boundaries of these uncertainties are not always known in advance, appropriate adaptive update laws are used to deal with them. Based on the Lyapunov stability theory and adaptive control method, some synchronization criteria on multiple sub-networks of complex network are established. Numerical simulations are given to demonstrate the feasibility and validity of the proposed scheme.

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1. Introduction

In the past few decades, many efforts have been devoted to the study of complex networks, in which the individuals in systems are denoted by the nodes of networks and the interactions between individuals are denoted by the links between nodes, such as the World Wide Web, communication networks, social networks, biological networks, power grid networks, and genetic regulatory networks [1–5]. Synchronization as one of the interesting and significant phenomena in complex networks, which not only can well explain many natural phenomena, but also has many potential applications in image processing and secure communication. Up to now, there are many widely-studied synchronization patterns, which define the correlated in-time behaviors among the nodes in dynamical networks, such as complete synchronization [6], impulsive synchronization [7], hybrid synchronization [8], lag synchronization [9], cluster synchronization [10], combination synchronization [11] and stochastic synchronization [12]. It should be noted that although there are so many results about the synchronization of complex networks, most of these works have been devoted to the research of the synchronization within

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a network [6,7,12–14] or the synchronization between two complex networks [8,9,15–17]. But in many cases, the only consideration of the synchronization within a network or even the synchronization between two networks cannot meet the needs of the reality. Recently, Wu et al. in [18] and Sun et al. in [19] have investigated the hybrid synchronization of two coupled complex dynamical networks with delayed and non-delayed coupling, each of which includes not only the inner synchronization within a network but also the outer synchronization between two complex networks. But the schemes proposed in this two papers are only applied to the synchronization of two coupled complex networks and each network is assumed to have the same number of nodes. However, the complex network usually consists of multiple sub-networks with non-identical nodes, and the number of nodes in different sub-networks is not always the same. Such as in the secure communication, in order to improve the security level of the transmitted signal, the transmitted signal may be split into several parts. Each part is loaded in different sub-networks respectively, these sub-networks with non-identical node dynamics and different number of nodes need to achieve the recovery of the information signal in a synchronous way.

Nevertheless, most of the previous papers on the synchronization of complex networks have just focused on the assumption that the nodes in drive-response networks have the same dimension. However, the synchronization between two or more networks with different dimensions of nodes is widespread in our reality, especially in biological science and social science. In [20], Dai et al. have investigated a generalized function projective lag synchronization of uncertain complex dynamical networks with different dimensions of nodes. The generalized outer synchronization between two completely different complex dynamical networks with different dimensions of nodes has been investigated in [21]. To the best of our knowledge, there are very few works on the synchronization of multiple networks with different dimensions of nodes. On the other hand, in the complex networks, the nodes not only within the same sub-network but also between different sub-networks need to interchange information with each other so as to accomplish a complex multi-function task collectively. Therefore, cooperation and competition play an indispensable role in realizing the synchronization of multiple sub-networks. The nodes in each sub-network may be endowed with different proportionality coefficients, which represent different proportions of the nodes in the process of completing the complex multi-function task. Combinatorial synchronization comes into being in just this context. The combinatorial synchronization taken as an extension of the traditional synchronization pattern, the primitive aim of which is to realize the synchronization between two drive systems and one response system, has drawn much attention from the researchers [22,23] due to its potential application in secure communication [24] and multiple-agent system [25,26]. However, almost all of the previous literature on the combinatorial synchronization have just focused on the synchronization of three or four chaotic systems [22,23]. More recently, Zhou et al. [11,24] have investigated the combinatorial synchronization of multiple sub-networks. But in these two papers, they only considered that, the nodes can only connect with each other within a sub-network and they cannot connect with the others belonging to different sub-networks at all. In addition, the boundaries of uncertainties for these nodes are all assumed to be known in advance, and the nodes in each sub-network are assumed to have the same dimension. However, in reality, the nodes in these sub-networks may need to interchange information with the others belonging to the same or even different sub-networks, and the boundaries of uncertainties for these nodes are not always known in advance. Moreover, the nodes belonging to different sub-networks may have different dynamical behaviors or even different dimensions.

In addition, complex networks with time-varying delays are more common because of the network traffic congestions as well as the finite speed of signal transmission. The adaptive complete synchronization between two neural networks with time-varying delay and distributed delay has been discussed in [27]. Some more synchronization criteria for the synchronization of complex networks with time-delays are derived in [28–31]. Furthermore, since noise and uncertainty factors are ubiquitous in the real systems, the synchronization of complex networks is unavoidably affected by all kinds of stochastic noise and uncertainty factors [7,8,11,12,15,17,20,29,31]. Effectively reflecting the influence of these stochastic and uncertain factors can help us recognize the real world reasonably. Motivated by the above discussions, in this paper, we aim to investigate the hybrid combinatorial synchronization on multiple dynamical sub-networks with different dimensions of nodes, where the nodes in each sub-network have different dynamical behaviors and all of them may be subject to some unknown boundaries of uncertainties and time-delays. By means of the Lyapunov stability theory and adaptive control method, some sufficient conditions on the hybrid combinatorial synchronization are proposed. Numerical simulations show the feasibility and effectiveness of the proposed scheme.

The rest of this paper is organized as follows. In Section 2, we introduce the detailed network structure model and provide some preliminary definitions and assumptions. Section 3 gives the stability analysis for the hybrid combinatorial synchronization on multiple sub-networks with unknown boundaries of uncertainties. In Section 4, numerical simulations are presented to verify the correctness and effectiveness of our theoretical results. The conclusion is finally drawn in Section 5.

Notation: $R^{n_{\mu_i}}$ denotes the n_{μ_i} -dimensional Euclidean space and $n_{\mu_i} \in N_+$. $\mathbf{0}$ and I represent a zero matrix and an identity matrix with suitable dimensions, respectively. The superscript “ T ” is the transpose. $\|\cdot\|$ stands for the mean square norm, which is defined as $\|e\| = (E\{e^T(t)e(t)\})^{1/2}$, where $E\{\cdot\}$ is the expected value function. For convenience, the nonlinear function $f(x(t))$ is equal to $f(x)$, error vector $e(t - \tau_t)$ is equal to e_{τ_t} , and time-varying delay $\tau(t)$ is equal to τ_t .

2. The description of the network model and some preliminaries

In this literature, we establish the following network model. Suppose that the complex network includes N nodes and it can be divided into m sub-networks G_1, G_2, \dots, G_m which depend on the local dynamics of nodes. Without loss of generality,

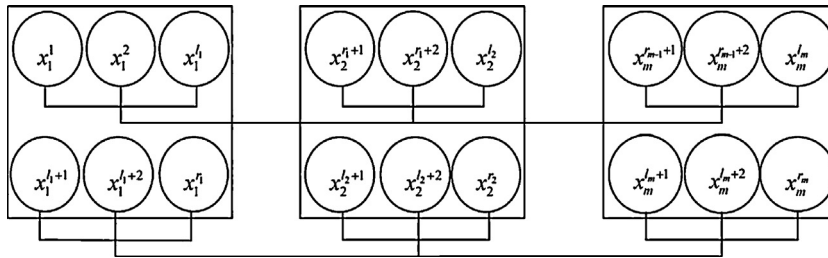


Fig. 1. The network structure diagram constructed by m star-like shape dynamical sub-networks with multiple preset-nodes in each dynamical sub-network.

the index set of all nodes in the k th dynamical sub-network can be described as $G_k = \{r_{k-1} + 1, \dots, r_k\}$ for $k \in \{1, 2, \dots, m\}$, where $r_0 = 0$, $r_m = N$ and $r_{k-1} < r_k$. In order to assign these N nodes into m sub-networks, we define an index function, namely $\mu : \{1, 2, \dots, N\} \rightarrow \{1, 2, \dots, m\}$. If node i belongs to the j th sub-network, then we have $\mu(i) = \mu_j = j$. The dynamical behaviors of the nodes belonging to the same sub-network are identical, and these nodes belonging to the same sub-network are connected by a star-like shape topological structure. It should be noted that the star-like shape topological structure in this paper is quite different from the general star topology proposed in Ref. [32]. In our proposed star-like shape topological structure, the nodes can be divided into two categories: general node and preset-node, where the preset-node taken as the response system in each sub-network needs applying controller. All the nodes with the same category can communicate with each other, while the nodes with different categories cannot communicate with the others at all. Moreover, the connections of the nodes are no longer restricted by the general star topology structure. The detailed network structure diagram consisting of m star-like shape dynamical sub-networks with multiple preset-nodes in each sub-network can be described in Fig. 1, where x_k^i ($i = r_{k-1} + 1, r_{k-1} + 2, \dots, l_k$) represents the general node i in the k th dynamical sub-network and $x_k^{i'}$ ($i' = l_k + 1, l_k + 2, \dots, r_k$) represents the preset-node i' in the k th dynamical sub-network, where $k = 1, 2, \dots, m$. That is to say, the general nodes in the k th dynamical sub-network can be marked as $r_{k-1} + 1, r_{k-1} + 2, \dots, l_k$, and the corresponding preset-nodes are marked as $l_k + 1, l_k + 2, \dots, r_k$. The line from x_k^i to $x_k^{i'}$ represents a connection between the nodes i and j in the k th dynamical sub-network.

The node dynamics of the general node with uncertainty in the complex network is characterized by

$$dx_i(t) = \left[f_{\mu_i}(x_i) + \Delta f_i(x_i) + \sum_{j=1}^N c_{ij}x_j(t - \tau_t) \right] dt, \quad i \in G/\mu_i \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, \dots, x_{in_{\mu_i}})^T \in R^{n_{\mu_i}}$ denotes the state vector of node i , $f_{\mu_i} : R^{n_{\mu_i}} \rightarrow R^{n_{\mu_i}}$ represents a nonlinear vector-valued function in the μ_i th dynamical sub-network, $\Delta f_i : R^{n_{\mu_i}} \rightarrow R^{n_{\mu_i}}$ represents the dynamical uncertainty of node i , τ_t is a time-varying delay. The matrix $C = (c_{ij})_{N \times N}$ is the zero-row-sum coupling configuration matrix, which represents the underlying topology structure of the complex network, and is defined as, if there is a connection between node i and node j ($j \neq i$), then $c_{ij} \neq 0$; otherwise, $c_{ij} = 0$. G/μ_i denotes the set of all the general nodes in the μ_i th dynamical sub-network.

The corresponding node dynamics of the preset-node with uncertainty and stochastic disturbance in the μ_i th dynamical sub-network can be denoted as follows

$$dx_i(t) = \left[f_{\mu_i}(x_i) + \Delta f_i(x_i) + \sum_{j=1}^N c_{ij}x_j(t - \tau_t) + u_i \right] dt + \delta(t, e_i(t), e_i(t - \tau_t))d\omega, \quad i \in \tilde{G}_{\mu_i} \tag{2}$$

where $u_i \in R^{n_{\mu_i}}$ is a designed controller in the μ_i th dynamical sub-network, $\delta : [0, +\infty) \times R^{n_{\mu_i}} \times R^{n_{\mu_i}} \rightarrow R^{n_{\mu_i} \times n_{\mu_i}}$ denotes the noise intensity function matrix, and $\omega(t)$ is an n_{μ_i} -dimensional Brownian motion defined on a complete probability space satisfying $E\{\delta d\omega(t)\} = 0$ and $E\{(\delta d\omega(t))^T(\delta d\omega(t))\} = \delta^T \delta dt$. \tilde{G}_{μ_i} denotes a node-set consisting of all the preset-nodes in the μ_i th dynamical sub-network, in which the nodes are in need of applying controllers for the combinatorial inner synchronization, and it satisfies the condition $G_{\mu_i} = G/\mu_i \cup \tilde{G}_{\mu_i}$. Moreover, $G = [G_1 \ G_2 \ \dots \ G_m]^T$ is a symmetric matrix due to the fact that the topological structure of the complex network we considered in this paper is nondirective.

Remark 1. As the fact that the stochastic communication noise only appears in the process of information transmission, in our proposed scheme, only the node as the response system in each sub-network needs adding the stochastic noise.

Now we give the definitions of the combinatorial inner synchronization within a dynamical sub-network and the combinatorial outer synchronization between multiple different dynamical sub-networks, and some other definitions for the matrix C .

Definition 1. The network (1) is said to reach combinatorial inner synchronization within the μ_i th dynamical sub-network if it satisfies the condition that $\lim_{t \rightarrow \infty} \left\| \sum_{i \in G_{\mu_i}} A_i x_i - \sum_{i \in \tilde{G}_{\mu_i}} A_i x_i \right\| = 0$, where $A_i (i \in G_{\mu_i})$ are constant diagonal matrices with suitable dimensions, and $A_i (i \in \tilde{G}_{\mu_i})$ are reversible.

Remark 2. The combinatorial synchronization is the extension of the traditional synchronization pattern, where the weighted matrix A_i represents the proportion of node i in the completion of a complex task. If there are two constants $\tilde{l}_1 \in G_{\mu_i}$ and $\tilde{l}_2 \in \tilde{G}_{\mu_i}$ such that $A_i = \begin{cases} \neq \mathbf{0}, & i = \tilde{l}_1 \\ \equiv \mathbf{0}, & i \neq \tilde{l}_1, \end{cases} i \in G_{\mu_i}$ and $A_i = \begin{cases} \neq \mathbf{0}, & i = \tilde{l}_2 \\ \equiv \mathbf{0}, & i \neq \tilde{l}_2, \end{cases} i \in \tilde{G}_{\mu_i}$, then the combinatorial synchronization becomes the modified projective synchronization; moreover, it becomes a complete synchronization when $A_i = \begin{cases} = I, & i = \tilde{l}_1 \\ \equiv \mathbf{0}, & i \neq \tilde{l}_1, \end{cases} i \in G_{\mu_i}$ and $A_i = \begin{cases} = I, & i = \tilde{l}_2 \\ \equiv \mathbf{0}, & i \neq \tilde{l}_2, \end{cases} i \in \tilde{G}_{\mu_i}$.

Let synchronization error in the μ_i th dynamical sub-network be $e_{\mu_i}(t) = \sum_{i \in G_{\mu_i}} A_i x_i - \sum_{i \in \tilde{G}_{\mu_i}} A_i x_i$, according to the drive system (1) and response system (2), the combinatorial inner synchronization error system within the μ_i th dynamical sub-network can be derived as

$$de_{\mu_i}(t) = \sum_{i \in G_{\mu_i}} A_i \left[f_{\mu_i}(x_i) + \Delta f_i(x_i) + \sum_{j=1}^N c_{ij} x_j(t - \tau_t) \right] dt - \sum_{i \in \tilde{G}_{\mu_i}} A_i \left[(f_{\mu_i}(x_i) + \Delta f_i(x_i) + \sum_{j=1}^N c_{ij} x_j(t - \tau_t) + u_i) dt + \delta(t, e_i, e_{i\tau_t}) d\omega \right] \tag{3}$$

Thus, our objective is to design a suitable controller u_i such that the error dynamical system (3) is asymptotically stable, i.e. $\lim_{t \rightarrow \infty} e_{\mu_i}(t) = 0, i \in \{1, 2, \dots, N\}$, which implies that all of the nodes within the μ_i th dynamical sub-network have realized the combinatorial inner synchronization.

When all the nodes belonging to the same dynamical sub-network have reached the combinatorial inner synchronization; and then, through a suitable switch action, the preset-nodes will turn to reach the combinatorial outer synchronization between different sub-networks. The drive and response systems in different dynamical sub-networks can be described as

$$dx_i(t) = \left[f_{\mu_i}(x_i) + \Delta f_i(x_i) + \sum_{j=1}^m \tilde{c}_{ij} x_j(t - \tau_t) \right] dt, \quad i \in \tilde{G}' \tag{4}$$

$$dx_i(t) = \left[f_{\mu_i}(x_i) + \Delta f_i(x_i) + \sum_{j=1}^m \tilde{c}_{ij} x_j(t - \tau_t) + \tilde{u}_i \right] dt + \tilde{\delta}(t, e_i, e_{i\tau_t}) d\omega, \quad i \in \tilde{G}'' \tag{5}$$

where $x_i = [x_{i1}, x_{i2}, \dots, x_{in_{\mu_i}}]^T \in R^{n_{\mu_i}}$ is the state vector of node $i (i \in \tilde{G}' \cup \tilde{G}'')$. \tilde{G}', \tilde{G}'' represent the node-sets consisting of the general nodes and the preset-nodes in the stage of the combinatorial outer synchronization, respectively, although they have served as the preset-nodes in the stage of the combinatorial inner synchronization; and they satisfy the condition $\tilde{G}' \cup \tilde{G}'' = \cup_{k=1}^m \tilde{G}_k$. $f_{\mu_i}(x_i) = [f_{\mu_i}(x_{i1}), f_{\mu_i}(x_{i2}), \dots, f_{\mu_i}(x_{in_{\mu_i}})]^T \in R^{n_{\mu_i}}$ is the continuous nonlinear vector function of node i in the μ_i th dynamical sub-network, and $\Delta f_i(x_i) \in R^{n_{\mu_i}}$ is the vector of uncertainty of node i . $\tilde{C} \in R^{m \times m}$ is a zero-row-sum connection matrix, whose elements \tilde{c}_{ij} are the connection weight between the preset-nodes i and j . $\tilde{u}_i \in R^{n_{\mu_i}}$ is a controller to be designed in \tilde{G}'' for the combinatorial outer synchronization. $\tilde{\delta} : [0, +\infty) \times R^{n_{\mu_i}} \times R^{n_{\mu_i}} \rightarrow R^{n_{\mu_i}} \times n_{\mu_i}$ denotes the noise intensity function.

Definition 2. If there are some constant matrices $\tilde{A}_i (i \in \tilde{G}' \cup \tilde{G}'')$ with suitable dimensions, and $\tilde{A}_i (i \in \tilde{G}'')$ are reversible, such that $\lim_{t \rightarrow \infty} \left\| \sum_{i \in \tilde{G}'} \tilde{A}_i x_i - \sum_{i \in \tilde{G}''} \tilde{A}_i x_i \right\| = 0$, then the drive system (4) and response system (5) are called to realize the combinatorial outer synchronization between multiple dynamical sub-networks.

Let $\tilde{e}(t) = \sum_{i \in \tilde{G}'} \tilde{A}_i x_i - \sum_{i \in \tilde{G}''} \tilde{A}_i x_i$ be the error state, the combinatorial outer synchronization error system between multiple dynamical sub-networks can be described as

$$d\tilde{e}(t) = \sum_{i \in \tilde{G}'} \tilde{A}_i \left[f_{\mu_i}(x_i) + \Delta f_i(x_i) + \sum_{j=1}^m \tilde{c}_{ij} x_j(t - \tau_t) \right] dt - \sum_{i \in \tilde{G}''} \tilde{A}_i \left\{ \left[f_{\mu_i}(x_i) + \Delta f_i(x_i) + \sum_{j=1}^m \tilde{c}_{ij} x_j(t - \tau_t) + \tilde{u}_i \right] dt + \tilde{\delta}(t, \tilde{e}, \tilde{e}_{\tau_t}) d\omega \right\} \tag{6}$$

In order to realize the combinatorial outer synchronization between multiple dynamical sub-networks, suitable controller \tilde{u}_i should be designed such that the error dynamical system (6) is asymptotically stable, i.e. $\lim_{t \rightarrow \infty} \tilde{e}(t) = 0$, which means that all of the preset-nodes between multiple dynamical sub-networks have reached the combinatorial outer synchronization.

Definition 3. Matrix $C=(c_{ij}) \in R^{(r_k-r_{k-1}) \times (r_k-r_{k-1})} (k=1, 2, \dots, m)$ is said to belong to class B_1 , denoted as $C \in B_1$, if $c_{ij} = c_{ji} \leq 0, i \neq j, c_{ii} = -\sum_{j=r_{k-1}+1, j \neq i}^{r_k} c_{ij}, i \in G'_k$ and $c_{ij} = \begin{cases} 0, & i \in \tilde{G}_k, i \in G'_k \\ \text{aconstant}, & i, j \in \tilde{G}_k \end{cases}$.

Definition 4. Matrix $C=(c_{ij}) \in R^{(r_k-r_{k-1}) \times (r_q-r_{q-1})} (k, q=1, 2, \dots, m)$ is said to belong to class B_2 , denoted as $C \in B_2$, if $\sum_{j=r_{q-1}+1}^{r_q} c_{ij} = 0, i \in G'_k$ and $c_{ij} = \begin{cases} 0, & i \in \tilde{G}_k, j \in G'_q \\ \text{aconstant}, & i \in \tilde{G}_k, j \in \tilde{G}_q \end{cases}$.

Definition 5. Suppose that the coupling matrix $C=(c_{ij}) \in R^{N \times N}$ of the complex network satisfies the following form:

$$C = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1m} \\ C_{21} & C_{22} & \dots & C_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & \dots & C_{mm} \end{pmatrix}$$

where $C_{kk} \in R^{(r_k-r_{k-1}) \times (r_k-r_{k-1})}$ belongs to B_1 and $C_{kq} \in R^{(r_k-r_{k-1}) \times (r_q-r_{q-1})} (k, q=1, 2, \dots, m)$ belongs to B_2 , then we say $C \in B_3$.

Remark 3. The specific process for reaching the hybrid combinatorial synchronization on multiple sub-networks of complex network can be simply described as follows:

- (a) The realization of the combinatorial inner synchronization within a sub-network. In this stage, the nodes in each sub-network are assigned with different proportionality coefficients, which represent different proportions in completing a complex multi-function task. Only the preset-nodes in each sub-network are in need of applying controllers. Moreover, the nodes in each sub-network can realize the combinatorial inner synchronization independently although there are some connections between the preset-nodes in different sub-networks.
- (b) Design a suitable time switch. As the fact that the synchronization time is computable and finite, we can design a suitable time switch, and the preset-nodes will turn to reach the combinatorial outer synchronization between different sub-networks after the predetermined time.
- (c) The realization of the combinatorial outer synchronization between different sub-networks. In this stage, the preset-nodes in the first $m - 1$ sub-networks serve as the new general nodes, and only the preset-nodes in the m th sub-network still serve as the new preset-nodes.

Prior to designing the synchronizing controllers, some assumptions must be noted.

A1. The unknown boundaries of uncertainties $\Delta f_i(x) (i=1, 2, \dots, N)$ are all norm-bounded, which means that there are some positive constants m_{μ_i} and m^* , such that $\left\| \sum_{i \in G'_{\mu_i}} A_i \Delta f_i(x_i) - \sum_{i \in \tilde{G}_{\mu_i}} A_i \Delta f_i(x_i) \right\| \leq m_{\mu_i}$ for the combinatorial inner synchronization within the μ_i th sub-network and $\left\| \sum_{i \in \tilde{G}} A_i \Delta f_i(x_i) - \sum_{i \in \tilde{G}'} A_i \Delta f_i(x_i) \right\| \leq m^*$ for the combinatorial outer synchronization between multiple sub-networks, while these boundaries are not always known in advance.

A2. δ is locally Lipschitz continuous and satisfies the linear growth condition [33] such that there exist two nonnegative constants p and q , satisfying the condition

$$\text{trace}((C\delta(t, e, e_{\tau_t}))^T C\delta(t, e, e_{\tau_t})) \leq \frac{p}{2} e^T e + \frac{q}{2} e_{\tau_t}^T e_{\tau_t}, \quad \forall (t, e, e_{\tau_t}) \in [0, +\infty) \times R^n \times R^n$$

Moreover, $\delta(t, 0, 0) \equiv 0$.

A3. $\tau(t)$ is a differentiable function with $0 \leq \dot{\tau}(t) \leq \varepsilon < 1$. Clearly, this assumption is justified when $\tau(t)$ is a constant.

Barbalat lemma [34]. If $w : R_+ \rightarrow R_+$ is a uniformly continuous function for $t \geq 0$ and if the limit of the integral $\lim_{t \rightarrow \infty} \int_0^t w(\lambda) d\lambda$ exists and is finite, then $\lim_{t \rightarrow \infty} w(t) = 0$.

3. Main results for the hybrid combinatorial synchronization on multiple dynamical sub-networks with uncertainties

In this section, we will propose some synchronization criteria for the combinatorial inner synchronization in a sub-network with unknown boundary of uncertainty, and the combinatorial outer synchronization between multiple dynamical sub-networks with different dimensions of nodes. In view of “birds of a feather flock together”, the complex network can be divided into multiple sub-networks such that the nodes belonging to the same dynamical sub-network are identical, while the ones belonging to different sub-networks are non-identical. With the network structure being more and more complex,

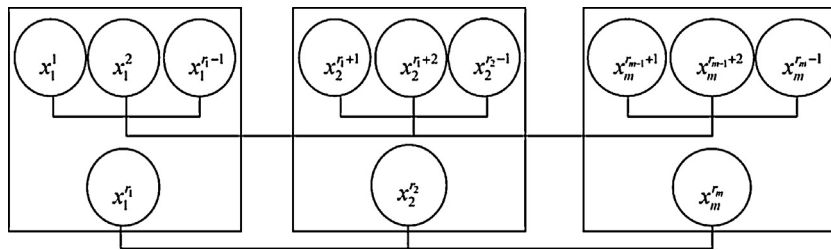


Fig. 2. The network structure diagram constructed by m star-like shape sub-networks with only one preset-node in each sub-network.

the study on the collective behaviors of the nodes within a sub-network and between multiple dynamical sub-networks becomes particularly important. In this paper, we take the star-like shape topological structure as the network model, and the k th dynamical sub-network has $\gamma_k = r_k - r_{k-1}$ ($k = 1, 2, \dots, m$) nodes. Furthermore, we label these nodes orderly; and for convenience, we just assume that the last node in each dynamical sub-network is selected as the sole preset-node. Therefore, the network structure diagram can be simplified in Fig. 2.

Remark 4. Fig. 2 is a special case of Fig. 1, however, if there are many preset-nodes in each sub-network, we can take the average state of these preset-nodes as the sole virtual preset-node that needs applying controller, just as the scheme proposed in Ref. [35].

3.1. The stability analysis for the combinatorial inner synchronization within the k th dynamical sub-network

Theorem 1. For the given scaling matrices $A_1, A_2, \dots, A_{\gamma_k}$, where A_{γ_k} is reversible, and $C \in B_3$, then the complex dynamical networks (1) and (2) can realize the combinatorial inner synchronization via the control law as below:

$$u_k(t) = A_{\gamma_k}^{-1} \left(\sum_{i=1}^{\gamma_k-1} A_i f_k(x_i) - A_{\gamma_k} f_k(x_{\gamma_k}) + \hat{m}_k \frac{e_k}{\|e_k\|} + \hat{\lambda}_k e_k + \sum_{i=1}^{\gamma_k-1} A_i \sum_{j=1}^N c_{ij} x_j(t - \tau_t) - A_{\gamma_k} \sum_{j=1}^N c_{\gamma_k j} x_j(t - \tau_t) \right) \tag{7}$$

and the corresponding adaptive update laws can be described as

$$\dot{\hat{m}}_k = k_1 \|e_k\|, \quad \dot{\hat{\lambda}}_k = k_2 e_k^T e_k \tag{8}$$

where k_1, k_2 are all arbitrary positive constants, and $k = 1, 2, \dots, m$.

Proof. Construct the Lyapunov function

$$V_k(t) = \frac{1}{2} e_k^T e_k + \frac{1}{2k_1} (m_k - \hat{m}_k)^2 + \frac{1}{2k_2} (\lambda_k - \hat{\lambda}_k)^2 + \frac{q}{4(1-\varepsilon)} \int_{t-\tau_t}^t e_k^T(s) e_k(s) ds \tag{9}$$

where m_k, λ_k are positive constants, taking derivative on both sides of (9) along the trajectories of (3), we have

$$dV_k(t) = E \left\{ e_k^T \left[\sum_{i=1}^{\gamma_k-1} A_i (f_k(x_i) + \Delta f_i(x_i)) + \sum_{j=1}^N c_{ij} x_j(t - \tau_t) - A_{\gamma_k} (f_k(x_{\gamma_k}) + \Delta f_{\gamma_k}(x_{\gamma_k})) + \sum_{j=1}^N c_{\gamma_k j} x_j(t - \tau_t) + u_k \right] \right. \\ \left. + \frac{1}{2} \text{trace}((A_{\gamma_k} \delta)^T (A_{\gamma_k} \delta)) + \frac{1}{k_1} (m_k - \hat{m}_k) (-\dot{\hat{m}}_k) + \frac{1}{k_2} (\lambda_k - \hat{\lambda}_k) (-\dot{\hat{\lambda}}_k) + \frac{q}{4(1-\varepsilon)} [e_k^T e_k - (1 - \dot{\tau}_t) e_{k\tau_t}^T e_{k\tau_t}] \right\} dt \tag{10}$$

Substituting (7) and (8) into (10), this yields

$$dV_k(t) = E \left\{ e_k^T \left[\sum_{i=1}^{\gamma_k-1} A_i \Delta f_i(x_i) - A_{\gamma_k} \Delta f_{\gamma_k}(x_{\gamma_k}) - \hat{m}_k \frac{e_k}{\|e_k\|} - \hat{\lambda}_k e_k \right] + \frac{1}{2} \text{trace}((A_{\gamma_k} \delta)^T (A_{\gamma_k} \delta)) + \frac{1}{k_1} (m_k - \hat{m}_k) (-k_1 \|e_k\|) \right. \\ \left. + \frac{1}{k_2} (\lambda_k - \hat{\lambda}_k) (-k_2 e_k^T e_k) + \frac{q}{4(1-\varepsilon)} [e_k^T e_k - (1 - \dot{\tau}_t) e_{k\tau_t}^T e_{k\tau_t}] \right\} dt$$

$$\begin{aligned}
 &= E \left\{ e_k^T \left[\sum_{i=1}^{\gamma_k-1} A_i \Delta f_i(x_i) - A_{\gamma_k} \Delta f_{\gamma_k}(x_{\gamma_k}) - \hat{m}_k \frac{e_k}{\|e_k\|} - \hat{\lambda}_k e_k \right] + \frac{1}{2} \text{trace}((A_{\gamma_k} \delta)^T (A_{\gamma_k} \delta)) - (m_k - \hat{m}_k) \|e_k\| \right. \\
 &\quad \left. - (\lambda_k - \hat{\lambda}_k) e_k^T e_k + \frac{q}{4(1-\varepsilon)} [e_k^T e_k - (1 - \tau_t) e_{k\tau_t}^T e_{k\tau_t}] \right\} dt \tag{11}
 \end{aligned}$$

In view of A1, A2 and A3, one can obtain

$$\begin{aligned}
 \dot{V}_k(t) &\leq \left\| \sum_{i=1}^{\gamma_k-1} A_i \Delta f_i(x_i) - A_{\gamma_k} \Delta f_{\gamma_k}(x_{\gamma_k}) \right\| \|e_k^T\| - \hat{m}_k \|e_k\| - \hat{\lambda}_k e_k^T e_k + \frac{p}{4} e_k^T e_k + \frac{q}{4} e_{k\tau_t}^T e_{k\tau_t} - (m_k - \hat{m}_k) \|e_k\| \\
 &\quad - (\lambda_k - \hat{\lambda}_k) e_k^T e_k + \frac{q}{4(1-\varepsilon)} [e_k^T e_k - (1 - \tau_t) e_{k\tau_t}^T e_{k\tau_t}] \leq (m_k - \hat{m}_k) \|e_k\| - \hat{\lambda}_k e_k^T e_k + \frac{p}{4} e_k^T e_k + \frac{q}{4} e_{k\tau_t}^T e_{k\tau_t} \\
 &\quad - (m_k - \hat{m}_k) \|e_k\| - (\lambda_k - \hat{\lambda}_k) e_k^T e_k + \frac{q}{4(1-\varepsilon)} [e_k^T e_k - (1 - \varepsilon) e_{k\tau_t}^T e_{k\tau_t}] \leq \left(\frac{p}{4} + \frac{q}{4(1-\varepsilon)} - \lambda_k \right) e_k^T e_k \leq -e_k^T e_k \leq 0 \tag{12}
 \end{aligned}$$

where $\lambda_k \geq \frac{p}{4} + \frac{q}{4(1-\varepsilon)} + 1$. From the inequality (12), we have $\dot{V}_k(t) < 0$. According to the Lasalle’s invariance principle, we can easily get the largest invariant set $E = \{e_k(t) \rightarrow 0, m_k \rightarrow \hat{m}_k, \lambda_k \rightarrow \hat{\lambda}_k, k \in \{1, 2, \dots, m\}\}$ as $t \rightarrow \infty$. That is to say, the synchronization error system (3) in the k th dynamical sub-network is asymptotically stable. This completes the proof.

Remark 5. $C \in B_3$ represents that the information transmission between the nodes belonging to the same or different sub-networks satisfies the conservation of energy.

3.2. The stability analysis for the combinatorial outer synchronization between multiple dynamical sub-networks with different dimensions of nodes

After all the nodes in each sub-network have realized the combinatorial inner synchronization, it will turn to the combinatorial outer synchronization between the preset-nodes in different dynamical sub-networks just by a switch action. Then we can build the combinatorial outer synchronization criterion between the drive system (4) and the response system (5).

Theorem 2. For the combinatorial outer synchronization between multiple dynamical sub-networks, the controller $\tilde{u}(t)$ can be designed as

$$\tilde{u}(t) = \tilde{A}_m^{-1} \left[\sum_{i=1}^{m-1} \tilde{A}_i f_i(x_{\gamma_i}) - \tilde{A}_m f_m(x_{\gamma_m}) + \hat{m} \frac{\tilde{e}}{\|\tilde{e}\|} + \hat{\lambda} \tilde{e} + \sum_{j=1}^m \left(\sum_{i=1}^{m-1} \tilde{A}_i \tilde{c}_{ij} x_j(t - \tau_t) - \tilde{A}_m \tilde{c}_{mj} x_j(t - \tau_t) \right) \right] \tag{13}$$

and the corresponding adaptive update laws can be selected as

$$\dot{\hat{m}} = k_3 \|\tilde{e}\|, \quad \dot{\hat{\lambda}} = k_4 \tilde{e}^T \tilde{e} \tag{14}$$

where k_3, k_4 are all arbitrary positive constants. Then, the combinatorial outer synchronization between multiple sub-networks is reached, and the synchronization error system (6) is globally asymptotically stable. The proof of Theorem 2 is similar to that of Theorem 1, which is omitted here.

Remark 6. The proposed scheme can be applied to not only the undirected complex network but also the directed complex network, only with the condition that each node satisfies the conservation of energy.

4. Numerical simulations

In the following section, we will give two numerical examples to demonstrate the effectiveness of the theoretic results on the hybrid combinatorial synchronization of multiple dynamical sub-networks. Just as what we know, not all the nodes are identical since some complex networks may consist of different types of nodes in reality, such as the information transmission in the human brain. Each neuron with different node dynamics is assigned to accomplish a part of the complex instruction with the others synergistically, and different weighting coefficients of the nodes may represent different proportions in completing the instruction transmission. Considering the complexity of large scale network, we just assume that the completion of a complex instruction requires the cooperation of three sub-networks, and we may as well set $\gamma_1 = 4, \gamma_2 = 4, \gamma_3 = 3$, thus the network size is $N = 11$. Numerical simulation results for the hybrid combinatorial synchronization on multiple dynamical sub-networks are shown the feasibility and validity of the proposed scheme. Three four-wing chaotic or hyper-chaotic systems [36–38] are taken as the node dynamics for these dynamical sub-networks. The corresponding dynamic equations can be given as follows.

The four-wing hyper-chaotic system [36] is considered as the node dynamics of the first sub-network:

$$\begin{cases} \dot{x} = a_1x - yz + w \\ \dot{y} = xz - b_1y \\ \dot{z} = xy - c_1z + xw \\ \dot{w} = -y \end{cases} \quad (15)$$

when the system parameters are selected as $a_1 = 8$, $b_1 = 40$, $c_1 = 14.9$, the system (15) can exhibit four-wing hyper-chaotic behavior.

The four-wing chaotic system [37] is considered as the node dynamics of the second sub-network:

$$\begin{cases} \dot{x} = a_2(y - x) + f_2yz \\ \dot{y} = c_2x + d_2y - xz \\ \dot{z} = -b_2z + xy \end{cases} \quad (16)$$

when the system parameters are selected as $a_2 = 14$, $b_2 = 43$, $c_2 = -1$, $d_2 = 16$, $f_2 = 4$, the four-wing system is chaotic.

The four-wing hyper-chaotic system [38] is considered as the node dynamics of the third sub-network:

$$\begin{cases} \dot{x} = a_3x - yz + f_3w \\ \dot{y} = -b_3y + xz \\ \dot{z} = -c_3z + xy + d_3x \\ \dot{w} = -h_3x \end{cases} \quad (17)$$

when the system parameters are selected as $a_3 = 4$, $b_3 = 12$, $c_3 = 5.5$, $d_3 = 1$, $f_3 = 2.5$, $h_3 = 1$, the dynamical behavior of the system (17) is hyper-chaotic.

4.1. Combinatorial inner synchronization within a dynamical sub-network

For the combinatorial inner synchronization within a sub-network, the internal connection matrix $C \in B_3$ can be chosen as $C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$, where $C_{ii}(i=1, 2, 3)$ represents the star-like shape topological structure for the i th dynamical sub-

network, we can take them as $C_{11} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $C_{12} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $C_{21} = C_{12}^T$, $C_{13} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$,
 $C_{31} = C_{13}^T$, $C_{22} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $C_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$, $C_{32} = C_{23}^T$, $C_{33} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. In this paper, sine or

cosine function with cyclical nature is selected as the unknown bounded function. According to the detailed theory analysis presented in Section 3, the combinatorial inner synchronization within a sub-network is easily realized. In the numerical simulation process, we take the matrices as $A_1 = \text{diag}(5, 2, 4, 3)$, $A_2 = \text{diag}(3, 5, 2, 4)$, $A_3 = \text{diag}(4, 2, 5, 3)$, $A_4 = \text{diag}(3, 2, 4, 3)$. $\delta(t, e, e_\tau) = \text{diag}\{\varpi_1(e_1 - e_{1\tau}), \varpi_2(e_2 - e_{2\tau}), \varpi_3(e_3 - e_{3\tau}), \varpi_4(e_4 - e_{4\tau})\}$ and the intensity factors are selected as $\varpi_1 = 5$, $\varpi_2 = -2$, $\varpi_3 = -4$, $\varpi_4 = 3$. The time-varying delay is selected as $\tau_t = e^t / (8 + 8e^t)$. The corresponding time evolution of the synchronization errors for the three different dynamical sub-networks can be depicted in Fig. 3. From this figure, it can be seen that the combinatorial inner synchronization errors $e_{11}, e_{12}, e_{13}, e_{14}$ within the first dynamical sub-network (the solid lines), e_{21}, e_{22}, e_{23} within the second dynamical sub-network (the dashdot lines), and $e_{31}, e_{32}, e_{33}, e_{34}$ within the third dynamical sub-network (the dash lines) all converge to zero promptly. The corresponding adaptive unknown boundaries of uncertainties and the feedback gains for these three different dynamical sub-networks are depicted in Fig. 4, which means that the realization of the combinatorial inner synchronization has good immunity to the influence of both the unknown boundaries of uncertainties and stochastic disturbances.

4.2. Combinatorial outer synchronization between multiple dynamical sub-networks with different dimensions of nodes

From Section 4.1, we know that all the nodes within a sub-network have synchronized, while in the information transmission, the completion of an instruction needs several sub-networks to coordinate with each other. The nodes belonging to different sub-networks may have different dynamic behaviors or even different dimensions, and these nodes may be

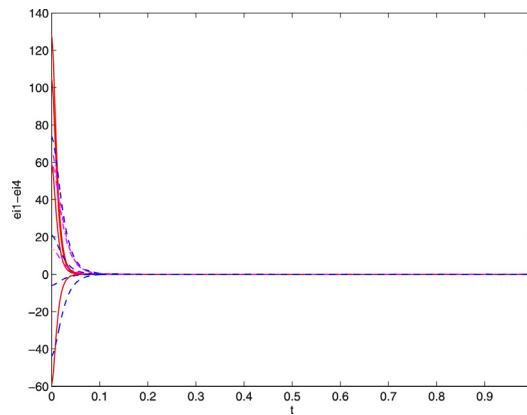


Fig. 3. The time evolution of the inner synchronization errors within three sub-networks.

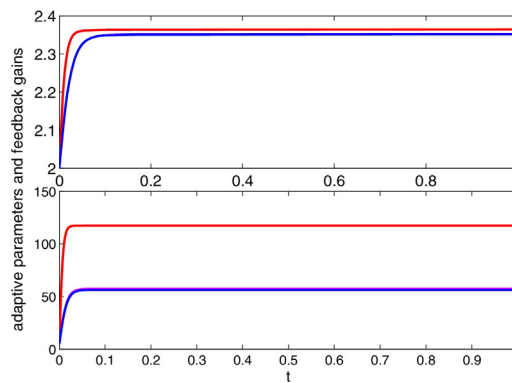


Fig. 4. The adaptive unknown boundary of uncertainties and feedback gains within three sub-networks.

disturbed by some stochastic uncertainties. In the following combinatorial outer synchronization between multiple dynamical sub-networks, the complete connection matrix is chosen as the exterior connection matrix, which can be written as

$$\tilde{C} = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{pmatrix} \tag{18}$$

The drive systems and the response system can be selected as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \\ \dot{w}_1 \end{pmatrix} = \underbrace{\begin{pmatrix} a_1x_1 - y_1z_1 + w_1 \\ x_1z_1 - b_1y_1 \\ x_1y_1 - c_1z_1 + x_1w_1 \\ -y_1 \end{pmatrix}}_{f_1(x_1)} + \underbrace{\begin{pmatrix} 0.1 \sin(\pi x_1) \\ 0.2 \sin(\pi y_1) \\ 0.3 \sin(\pi z_1) \\ 0.4 \sin(\pi w_1) \end{pmatrix}}_{\Delta f_1(x_1)} + \underbrace{\begin{pmatrix} x_1(t-\tau) - 2x_2(t-\tau) + x_3(t-\tau) \\ y_1(t-\tau) - 2y_2(t-\tau) + y_3(t-\tau) \\ z_1(t-\tau) - 2z_2(t-\tau) + z_3(t-\tau) \\ w_1(t-\tau) - 2w_2(t-\tau) + w_3(t-\tau) \end{pmatrix}}_{\sum_{j=1}^3 \tilde{c}_{1j}x_j(t-\tau)} \tag{19}$$

$$\begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \\ \dot{z}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} a_2(y_2 - x_2) + f_2y_2z_2 \\ c_2x_2 + d_2y_2 - x_2z_2 \\ -b_2z_2 + x_2y_2 \end{pmatrix}}_{f_2(x_2)} + \underbrace{\begin{pmatrix} 0.1 \sin(\pi x_2) \\ 0.2 \sin(\pi y_2) \\ 0.3 \sin(\pi z_2) \end{pmatrix}}_{\Delta f_2(x_2)} + \underbrace{\begin{pmatrix} x_1(t-\tau) + x_2(t-\tau) - 2x_3(t-\tau) \\ x_1(t-\tau) + x_2(t-\tau) - 2x_3(t-\tau) \\ x_1(t-\tau) + x_2(t-\tau) - 2x_3(t-\tau) \end{pmatrix}}_{\sum_{j=1}^3 \tilde{c}_{2j}x_j(t-\tau)}$$

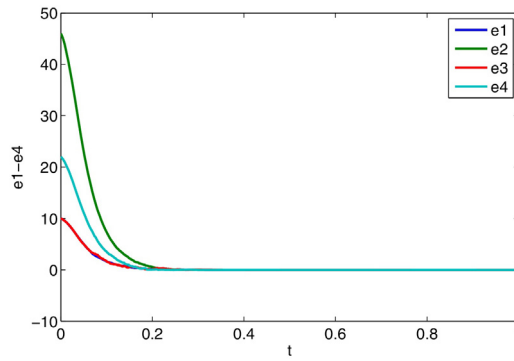


Fig. 5. The time evolution of the combinatorial outer synchronization errors between multiple dynamical sub-networks.

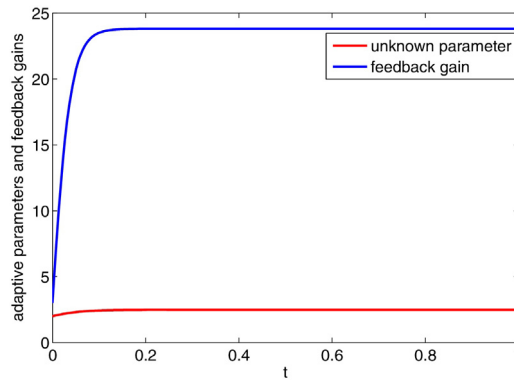


Fig. 6. The adaptive unknown boundary of uncertainty and feedback gain between multiple dynamical sub-networks.

$$\begin{aligned}
 \begin{pmatrix} dx_3 \\ dy_3 \\ dz_3 \\ dw_3 \end{pmatrix} &= \underbrace{\begin{pmatrix} a_3x_3 - y_3z_3 + f_3w_3 \\ -b_3y_3 + x_3z_3 \\ -c_3z_3 + x_3y_3 + d_3x_3 \\ -h_3x_3 \end{pmatrix}}_{f_3(x_3)} dt + \underbrace{\begin{pmatrix} 0.2 \cos(x_3) \\ -0.3 \sin(y_3) \\ 0.4 \cos(z_3) \\ 0.5 \sin(w_3) \end{pmatrix}}_{\Delta f_3(x_3)} dt + \underbrace{\begin{pmatrix} -2x_1(t - \tau_t) + x_2(t - \tau_t) + x_3(t - \tau_t) \\ -2y_1(t - \tau_t) + y_2(t - \tau_t) + y_3(t - \tau_t) \\ -2z_1(t - \tau_t) + z_2(t - \tau_t) + z_3(t - \tau_t) \\ -2w_1(t - \tau_t) + w_2(t - \tau_t) + w_3(t - \tau_t) \end{pmatrix}}_{\sum_{j=1}^3 \tilde{e}_{3j}x_j(t - \tau_t)} dt \\
 &+ \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}}_{\tilde{u}} dt + \underbrace{\begin{pmatrix} \varpi_1(e_1 - e_{1\tau_t}) & 0 & 0 & 0 \\ 0 & \varpi_2(e_2 - e_{2\tau_t}) & 0 & 0 \\ 0 & 0 & \varpi_3(e_3 - e_{3\tau_t}) & 0 \\ 0 & 0 & 0 & \varpi_4(e_4 - e_{4\tau_t}) \end{pmatrix}}_{\tilde{\delta}(t, e, e_{\tau_t})} \underbrace{\begin{pmatrix} d\omega_1 \\ d\omega_2 \\ d\omega_3 \\ d\omega_4 \end{pmatrix}}_{d\omega} \quad (20)
 \end{aligned}$$

During the simulation process, we take the matrices as $A_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$, $A_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \\ 2 & 0 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$. The

intensity factors are selected as $\varpi_1 = 5$, $\varpi_2 = -2$, $\varpi_3 = -8$, $\varpi_4 = 4$ and the time-varying delay is selected as $\tau_t = e^t / (5 + 5e^t)$. The time evolution of the synchronization errors is shown in Fig. 5. It is clear that the combinatorial outer synchronization errors e_1, e_2, e_3, e_4 converge to zero shortly. The adaptive unknown boundary of uncertainty and feedback weight are shown in Fig. 6, which means that the combinatorial outer synchronization between multiple dynamical sub-networks with different dimensions of nodes is realized even under the influence of unknown boundary of uncertainty and stochastic disturbance.

5. Conclusions

In this paper, based on the way of information processing on the neural network, a star-like shape network model is constructed to realize the hybrid combinatorial synchronization on multiple dynamical sub-networks with different dimensions of nodes and unknown boundaries of uncertainties. In the process of realizing the hybrid combinatorial synchronization, the nodes in each sub-network are assigned with different weighting coefficients, which represent different proportions in completing the instruction transmission. Only the nodes with the same category can communicate with each other, while the nodes with different categories cannot communicate with each other at all. The preset-nodes taken as the nodes in need of applying controllers, are assigned to coordinate with not only the nodes belonging to the same dynamical sub-network with the preset-node to realize the combinatorial inner synchronization, but also the other preset-nodes belonging to different dynamical sub-networks to realize the combinatorial outer synchronization. The conversion between the combinatorial inner synchronization and outer synchronization is implemented by a switch action. In addition, the nodes in a dynamical sub-network may be disturbed by some stochastic uncertainties, and the boundaries of these uncertainties are not always known in advance, suitable adaptive update laws are used to deal with them. According to the Lyapunov stability theory and adaptive control method, some sufficient conditions for the hybrid combinatorial synchronization on multiple dynamical sub-networks of complex network are obtained. Numerical simulations are provided to demonstrate the correctness and effectiveness of the derived theoretical results.

Compared with the previous literature, the main contribution of this paper lies in the study on the hybrid combinatorial synchronization of multiple sub-networks with different dimensions of nodes and unknown boundaries of uncertainties, which can be summarized as follows: (1) a star-like shape topological structure is proposed. Based on this topological structure, only a few controllers are in need of designing to realize the combinatorial inner synchronization within a sub-network and the combinatorial outer synchronization between multiple sub-networks with different dimensions of nodes; (2) the boundaries of uncertainties are not always known in advance, suitable adaptive update laws are used to deal with them; (3) both the cooperation and competition of the nodes are all considered, but the information transmission through each node satisfies the conservation of energy; (4) the proposed method is the extension of the method proposed in [11], and it can be applied to a more general complex network. In the future, we will investigate some other works on the multiple networks of complex network, such as the exponential synchronization, hybrid control, time-varying networks structures, and its application in other fields.

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